March 1 Math 2335 sec 51 Spring 2016

Section 4.1: Polynomial Interpolation

Context: We consider a set of distinct data points $\{(x_i, y_i) \mid i = 0, \ldots, n\}$ that we wish to fit with a polynomial curve.

- For a set of $n + 1$ points, we can fit a polynomial $P_n(x)$ of degree at most n.
- We assume that the points are distinct in the sense that $x_i \neq x_j$ when $i \neq j$.
- We will have two formulations, a Lagrange formulation and a Newton divided difference formulation.

Lagrange Interpolation Formula

Suppose we have $n+1$ distinct points $(x_0, y_0), (x_1, y_1), \ldots, (x_n, y_n)$. We define the $n+1$ Lagrange interpolation basis functions $L_0, L_1, \ldots, L_n$ by

$$L_i(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)(x_i-x_1)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

for $i = 0, \ldots, n$.

$$\text{Compactly:} \quad L_i(x) = \prod_{k=0, k\neq i}^{n} \left(\frac{x - x_k}{x_i - x_k} \right), \quad i = 0, \ldots, n$$

Lagrange's Formula The unique polynomial of degree $\leq n$ passing through these $n+1$ points is

$$P_n(x) = y_0 L_0(x) + y_1 L_1(x) + \cdots + y_n L_n(x).$$

Newton Divided Differences

Definition: Let f be a function whose domain contains the two distinct numbers x_0 and x_1. We define the *first-order divided difference* of $f(x)$ by

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

Notation: We'll use the square brackets "[ ]" with commas between the numbers to denote the divided difference.

Zeroth divided difference $\quad f[x_0] = f(x_0)$

Higher Order Divided Differences

Suppose we start with three distinct values x_0, x_1, x_2 in our domain. We can compute two first order divided differences

$$f[x_0, x_1] = \frac{f(x_1) - f(x_0)}{x_1 - x_0} \quad \text{and} \quad f[x_1, x_2] = \frac{f(x_2) - f(x_1)}{x_2 - x_1}.$$

Definition: The *second-order divided difference* of $f(x)$ at the points $x_0, x_1,$ and x_2 is

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0}.$$

Higher Order Divided Differences

Let $x_0, x_1, \ldots, x_n$ be distinct numbers in the domain of the function f.

Definition: The *third-order divided difference* of $f(x)$ at the points x_0, x_1, x_2, and x_3 is

$$f[x_0, x_1, x_2, x_3] = \frac{f[x_1, x_2, x_3] - f[x_0, x_1, x_2]}{x_3 - x_0}.$$

Definition: The n^{th}*-order divided difference* of $f(x)$ at the points $x_0, \ldots, x_n$ is

$$f[x_0, \ldots, x_n] = \frac{f[x_1, \ldots, x_n] - f[x_0, \ldots, x_{n-1}]}{x_n - x_0}.$$

Properties of Newton Divided Differences

Symmetry: Let $\{x_{i_0}, x_{i_1}, \ldots, x_{i_n}\}$ be any permutation (rearrangement) of the numbers $\{x_0, x_1, \ldots, x_n\}$. Then

$$f[x_{i_0}, x_{i_1}, \ldots, x_{i_n}] = f[x_0, x_1, \ldots, x_n].$$

(That is, the order of the x-values doesn't affect the value of the divided difference!)

Properties of Newton Divided Differences

Relation to Derivatives:

Theorem: Suppose f is n times continuously differentiable on an interval $\alpha \le x \le \beta$, and that $x_0, \ldots, x_n$ are distinct numbers in this interval. Then

$$f[x_0, x_1, \ldots, x_n] = \frac{1}{n!} f^{(n)}(c)$$

for some number c between the smallest and the largest of the numbers $x_0, \ldots, x_n$.

For example,

$$f[x_0, x_1] = f'(c), \quad f[x_0, x_1, x_2] = \frac{1}{2!} f''(c), \quad f[x_0, x_1, x_2, x_3] = \frac{1}{3!} f'''(c)$$

where in each case, c is some number between the least and greatest of the x_i values.

Interpolating Polynomial: Newton Divided Difference

Suppose we have $n+1$ distinct data points $(x_0, f(x_0))$, $(x_1, f(x_1)), \ldots, (x_n, f(x_n))$.

Linear Interpolation: The linear interpolating polynomial through $(x_0, f(x_0))$ and $(x_1, f(x_1))$ can be written as

$$P_1(x) = f(x_0) + (x - x_0) f[x_0, x_1].$$

Quadratic Interpolation: The quadratic interpolating polynomial through $(x_0, f(x_0))$, $(x_1, f(x_1))$, and $(x_2, f(x_2))$ can be written as

$$\begin{aligned} P_2(x) &= f(x_0) + (x - x_0) f[x_0, x_1] + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \\ &= P_1(x) + (x - x_0)(x - x_1) f[x_0, x_1, x_2] \end{aligned}$$

Interpolating Polynomial: Newton Divided Difference

Higher degree polynomials are defined recursively

Cubic Interpolation: The cubic interpolating polynomial through $(x_0, f(x_0))$, $(x_1, f(x_1))$, $(x_2, f(x_2))$, and $(x_3, f(x_3))$ can be written as

$$\begin{aligned} P_3(x) &= f(x_0) + (x - x_0)f[x_0, x_1] + (x - x_0)(x - x_1)f[x_0, x_1, x_2] + \\ &+ (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \\ \\ &= P_2(x) + (x - x_0)(x - x_1)(x - x_2)f[x_0, x_1, x_2, x_3] \end{aligned}$$

Interpolating Polynomial: Newton Divided Difference Formula

k^{th} **Degree Interpolation:** For $k \geq 2$, the polynomial of degree at most k through the points $(x_0, f(x_0)), \ldots (x_k, f(x_k))$ is

$$P_k(x) = P_{k-1}(x) + (x - x_0)(x - x_1) \cdots (x - x_{k-1}) f[x_0, \ldots, x_k]$$

Example

Consider the function $f(x) = 1/(1+x)$ and let $x_0 = 0$, $x_1 = 1$ and $x_2 = 2$.

(a) Compute the divided differences $f[x_0, x_1]$ and $f[x_0, x_1, x_2]$.

$$f(x_0) = f(0) = \frac{1}{1+0} = 1, \quad f(x_1) = f(1) = \frac{1}{1+1} = \frac{1}{2}, \text{ and}$$

$$f(x_2) = f(2) = \frac{1}{1+2} = \frac{1}{3}$$

$$f[x_0, x_1] = \frac{f(1) - f(0)}{1-0} = \frac{\frac{1}{2} - 1}{1} = \frac{-1}{2}$$

$$f[x_1, x_2] = \frac{f(2) - f(1)}{2-1} = \frac{\frac{1}{3} - \frac{1}{2}}{1} = \frac{-1}{6}$$

$$f[x_0, x_1, x_2] = \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{\frac{-1}{6} - \left(\frac{-1}{2}\right)}{2} = \frac{1}{6}$$

Example Continued...

So

$$f[x_0, x_1] = \frac{-1}{2}$$

and

$$f[x_0, x_1, x_2] = \frac{1}{6}$$

Example Continued...

(b) Find the first and second degree interpolating polynomials P_1 and P_2 using the Newton divided difference formula.

$$P_1(x) = f(x_0) + (x - x_0) f[x_0, x_1]$$

$$P_1(x) = 1 + (x - 0)\left(\frac{-1}{2}\right) = \frac{-1}{2}x + 1$$

$$P_2(x) = P_1(x) + (x - x_0)(x - x_1) f[x_0, x_1, x_2]$$

$$P_2(x) = \frac{-1}{2}x + 1 + x(x-1)\left(\frac{1}{6}\right)$$

$$P_2(x) = \frac{-1}{2}x + 1 + \frac{1}{6}(x^2 - x)$$

$$P_2(x) = \frac{1}{6}x^2 - \frac{2}{3}x + 1$$

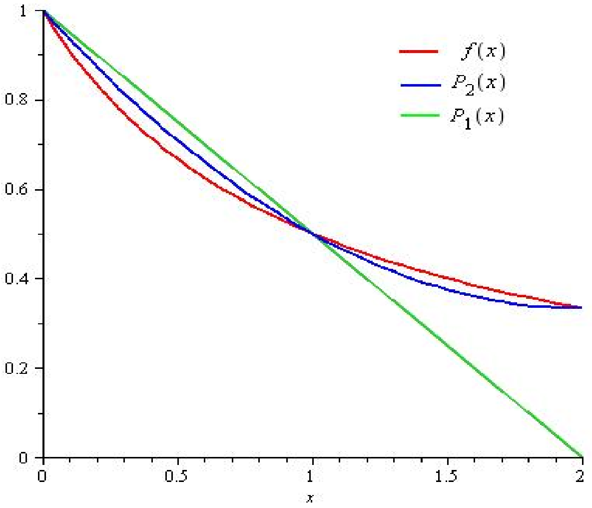

Figure: The function $f(x) = 1/(1+x)$ together with interpolating polynomials P_1 and P_2 using $x_0 = 0$, $x_1 = 1$, and $x_2 = 2$.

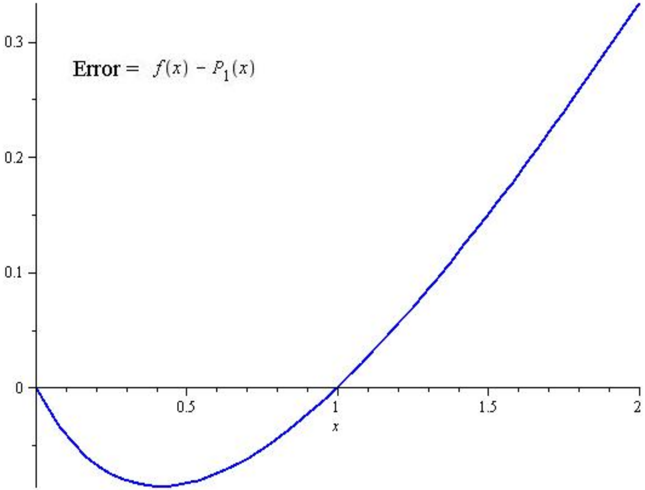

Figure: Error in the linear interpolation.

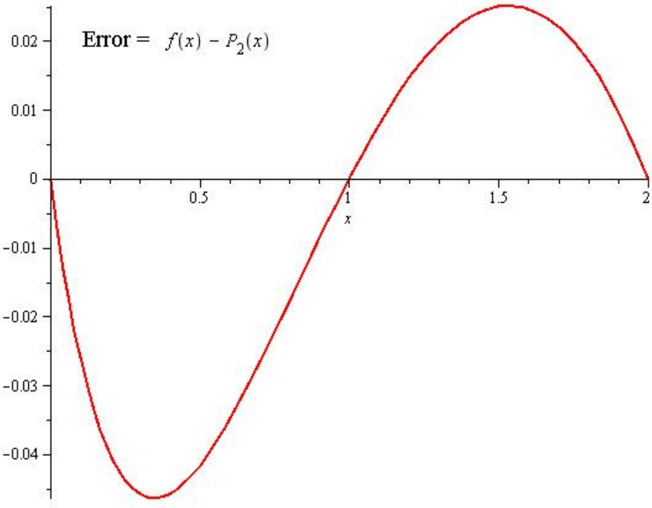

Figure: Error in the quadratic interpolation.

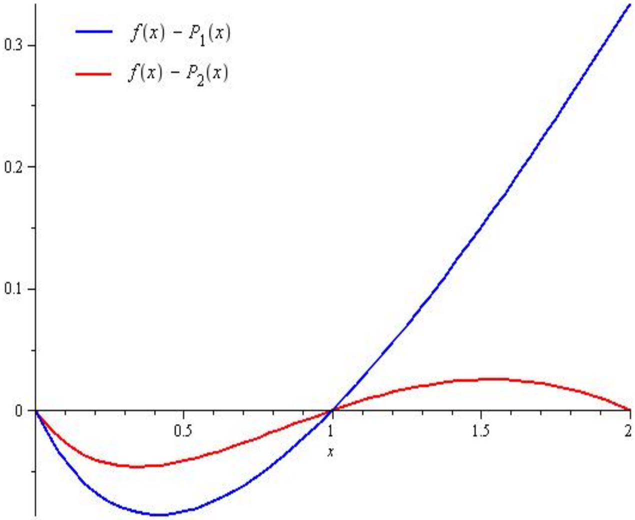

Figure: Comparison of errors when using P_1 versus P_2 to approximate f.

Section 4.2: Error in Polynomial Interpolation

The last example suggests that a higher degree polynomial results in *less error*. We'd like to characterize the error. It depends on the nature of the data (both the x and y-values).

Recall that we are interpolating data $(x_0, y_0), \ldots, (x_n, y_n)$—a.k.a. $(x_0, f(x_0)), \ldots, (x_n, f(x_n))$[1]—with the polynomial P_n of degree at most n given by

$$P_n(x) = \sum_{j=0}^{n} f(x_j) L_j(x).$$

We'll often call the numbers $x_0, \ldots, x_n$ **nodes.**

[1] $y_k = f(x_k)$

Theorem

Theorem: For $n \geq 0$, suppose f has $n+1$ continuous derivatives on $[a,b]$ and let $x_0, \ldots, x_n$ be distinct nodes in $[a,b]$. Then

$$f(x) - P_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(c_x)}{(n+1)!}$$

where c_x is some number between the smallest and largest values of $x_0, \ldots, x_n$ and x.

Note what this says: It says that

$$\text{Err}(P_n(x)) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(c_x)}{(n+1)!}$$

(The number c_x isn't known, but we can use this result to bound the error.)

Remark about the error formula

The error can be restated as

$$\text{Err}(P_n(x)) = \Psi_n(x)\ \frac{f^{(n+1)}(c_x)}{(n+1)!}$$

where Ψ_n is the $n+1$ degree **monic** polynomial[2]

$$\Psi_n(x) = (x - x_0)\cdots(x - x_n) = x^{n+1} + \text{terms with smaller powers}$$

The coefficients of those *smaller powers* depend on $x_0, \ldots, x_n$.

The error depends on the y's due to $f^{(n+1)}(c_x)$, and on the x's due to $\Psi_n(x)$.

[2] A monic polynomial is one whose leading coefficient is 1

Example

Take $f(x) = \sin x$ on the interval $[0, \frac{\pi}{2}]$. Let $0 \leq x_0 < x_1 \leq \frac{\pi}{2}$ and consider the linear interpolation $P_1(x)$. For $x_0 < x < x_1$, show that

$$|f(x) - P_1(x)| \leq \frac{h^2}{8} \quad \text{where} \quad h = x_1 - x_0$$

$$f(x) - P_1(x) = (x-x_0)(x-x_1)\frac{f''(c)}{2!}$$ for some c between x_0, x_1, and x

$f(x) = \sin x$

$f'(x) = \cos x$

$f''(x) = -\sin x$

For $0 \leq c \leq \pi/2$

$|-\sin c| \leq 1$

For $x_0 < x < x_1$, $x - x_0 > 0$, $x - x_1 < 0$

so $|(x-x_0)(x-x_1)| = (x-x_0)(x_1-x)$

A parabola open downward with maximum at the vertex where $x = \frac{x_0+x_1}{2}$

$$|\psi_1(x)| = |(x-x_0)(x_1-x)| \Rightarrow$$

$$\left|\psi_1\left(\frac{x_0+x_1}{2}\right)\right| = \left|\left(\frac{x_0+x_1}{2} - x_0\right)\left(x_1 - \frac{x_1+x_0}{2}\right)\right|$$

$$= \left| \left(\frac{x_1 - x_0}{2}\right)\left(\frac{x_1 - x_0}{2}\right) \right| = \frac{h^2}{4}$$

So maximum for $|\Psi_1(x)|$ is $\frac{h^2}{4}$ and for $|f''(c)|$ is 1.

$$|f(x) - P_1(x)| = \left| \Psi_1(x) \frac{f''(c)}{2!} \right| \le \frac{h^2}{4} \cdot \frac{1}{2} = \frac{h^2}{8}$$

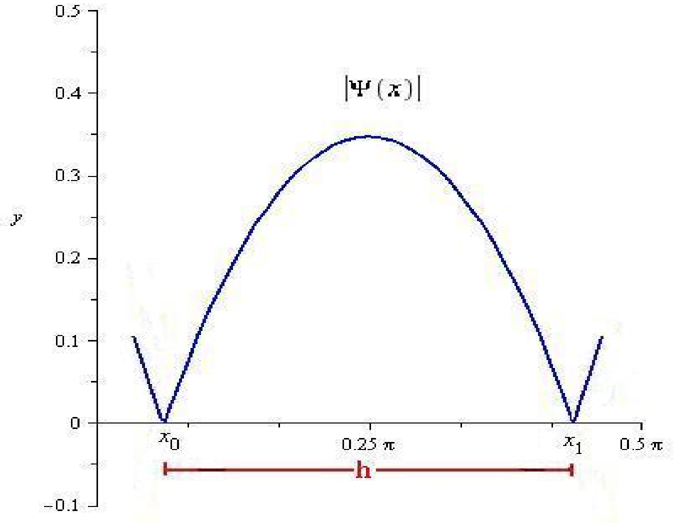

Figure: The maximum value of $(x - x_0)(x_1 - x)$ occurs at the vertex $\frac{x_0+x_1}{2}$.

Example

Again take $f(x) = \sin x$ on the interval $[0, \frac{\pi}{2}]$. Let $0 \le x_0 < x_1 < x_2 \le \frac{\pi}{2}$ and consider the quadratic interpolation $P_2(x)$. For $x_0 < x < x_2$, show that

$$|f(x) - P_2(x)| \le \frac{h^3}{9\sqrt{3}} \quad \text{where} \quad h = x_1 - x_0 = x_2 - x_1$$

$$f(x) - P_2(x) = (x-x_0)(x-x_1)(x-x_2)\frac{f'''(c)}{3!}$$

for some c between x_0 and x_2

$$f'''(x) = -\cos x$$

for $0 \le c \le \pi/2$

$$|-\cos c| \le 1$$

Here $\psi_2(x) = (x - x_0)(x - x_1)(x - x_2)$

Let $t = x - x_1$ Since $x_0 = x_1 - h$ $x - x_0 = x - (x_1 - h)$

$= t + h$

Since $x_2 = x_1 + h$, $x - x_2 = x - (x_1 + h)$

$= t - h$

So $\psi_2(t) = (t+h)t(t-h) = t^3 - h^2 t$

Use Calculus to find the maximum value of

$|\psi_2(t)|$

$$\psi_2'(t) = 3t^2 - h^2 \qquad \psi_2'(t) = 0 \Rightarrow 3t^2 - h^2 = 0$$

$$\Rightarrow t^2 = \frac{h^2}{3} \Rightarrow t = \frac{\pm h}{\sqrt{3}}$$

2nd derivative test $\psi_2''(t) = 6t$

$\psi_2''\left(\frac{-h}{\sqrt{3}}\right) = \frac{-6h}{\sqrt{3}} < 0$ local maximum.

maximum $|\psi_2(t)| = \left|\psi_2\left(\frac{-h}{\sqrt{3}}\right)\right| = \left|\frac{-h^3}{3\sqrt{3}} - h^2\left(\frac{-h}{\sqrt{3}}\right)\right|$

$$= \frac{h^3}{\sqrt{3}}\left(\frac{-1}{3} + 1\right) = \frac{2h^3}{3\sqrt{3}}$$

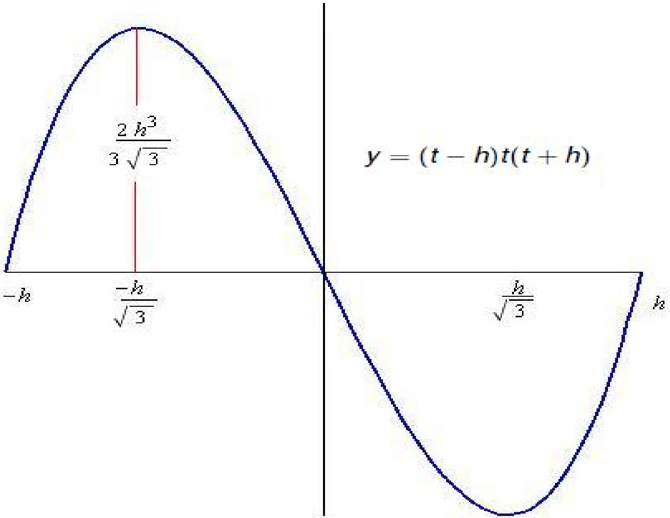

Figure: The maximum value of $|(t+h)t(t-h)|$ occurs at $\pm\frac{h}{\sqrt{3}}$.

So

maximum of $|\psi_2(x)|$ is $\frac{2h^3}{3\sqrt{3}}$

maximum of $|f'''(c)|$ is 1

$$|f(x) - P_2(x)| = |\psi_2(x)| \left| \frac{f'''(c)}{3!} \right| \leq \frac{2h^3}{3\sqrt{3}} \cdot \frac{1}{6}$$

$$= \frac{h^3}{9\sqrt{3}}$$

Example

Take $f(x) = \ln(x+4)$ on the interval $[-1, 1]$. Let $x_0 = -1$, $x_1 = 0$, $x_2 = 1$ and consider the quadratic interpolation $P_2(x)$. For $-1 < x < 1$, show that

$$|f(x) - P_2(x)| \leq \left(\frac{2}{27}\right)\left(\frac{1}{9\sqrt{3}}\right)$$

Here x_0, x_1, x_2 are equally spaced.

$x_1 - x_0 = 0 - (-1) = 1$ and $x_2 - x_1 = 1 - 0 = 1$

i.e. $h = 1$.

So $\quad |\Psi_2(x)| \leq \dfrac{2(1)^3}{3\sqrt{3}}$

$f(x) = \ln(x+4)$, $f''(x) = \frac{-1}{(x+4)^2}$

$f'(x) = \frac{1}{x+4}$, $f'''(x) = \frac{2}{(x+4)^3}$

f''' is decreasing and positive on $[-1,1]$. So it's biggest at the left end point -1.

$$|f'''(c)| \le \frac{2}{(-1+4)^3} = \frac{2}{3^3} = \frac{2}{27} \text{ for all } c \text{ in } [-1,1].$$

So

$$|f(x) - P_2(x)| = \left| \Psi_2(x) \frac{f'''(c)}{3!} \right|$$

$$\leq \frac{2}{3\sqrt{3}} \cdot \frac{2/27}{6} = \left(\frac{1}{9\sqrt{3}}\right)\left(\frac{2}{27}\right)$$