March 1 Math 3260 sec. 55 Spring 2018

Section 3.3: Crammer's Rule, Volume, and Linear Transformations

Crammer's Rule is a method for solving a square system $A\mathbf{x} = \mathbf{b}$ by use of determinants. While it is impractical for large systems, it provides a fast method for some small systems (say 2×2 or 3×3).

Definition: For $n \times n$ matrix A and $\mathbf{b}$ in $\mathbb{R}^n$, let $A_i(\mathbf{b})$ be the matrix obtained from A by replacing the i^{th} column with the vector $\mathbf{b}$. That is

$$A_i(\mathbf{b}) = [\mathbf{a}_1 \cdots \mathbf{a}_{i-1}\ \mathbf{b}\ \mathbf{a}_{i+1} \cdots \mathbf{a}_n]$$

Crammer's Rule

Theorem: Let A be an $n \times n$ nonsingular matrix. Then for any vector $\mathbf{b}$ in $\mathbb{R}^n$, the unique solution of the system $A\mathbf{x} = \mathbf{b}$ is given by $\mathbf{x}$ where

$$x_i = \frac{\det A_i(\mathbf{b})}{\det A}, \quad i = 1, \ldots, n$$

Example

Determine whether Crammer's rule can be used to solve the system. If so, use it to solve the system.

$$\begin{array}{rcrcr} 2x_1 & + & x_2 & = & 9 \\ -x_1 & + & 7x_2 & = & -3 \end{array}$$

$$\underbrace{\begin{bmatrix} 2 & 1 \\ -1 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\vec{x}} = \underbrace{\begin{bmatrix} 9 \\ -3 \end{bmatrix}}_{\vec{b}}$$

$\det(A) = 2 \cdot 7 - (-1) \cdot 1 = 15$ $\quad \det(A) \neq 0$ $\quad$ A is nonsingular

$$A_1(\vec{b}) = \begin{bmatrix} 9 & 1 \\ -3 & 7 \end{bmatrix}, \quad A_2(\vec{b}) = \begin{bmatrix} 2 & 9 \\ -1 & -3 \end{bmatrix}$$

$$\det(A_1(\vec{b})) = 9\cdot 7 - (-3)\cdot 1 = 63 + 3 = 66$$

$$\det(A_2(\vec{b})) = 2(-3) - (-1)\cdot 9 = -6 + 9 = 3$$

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{66}{15} = \frac{22}{5}$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{3}{15} = \frac{1}{5}$$

$$\vec{x} = \begin{bmatrix} \frac{22}{5} \\ \frac{1}{5} \end{bmatrix}$$

Example

Determine whether Crammer's rule can be used to solve the system. If so, use it to solve the system.

$$\begin{array}{rcrcrcl} x_1 & + & 2x_2 & + & 3x_3 & = & 3 \\ & & x_2 & + & 4x_3 & = & 3 \\ 5x_1 & + & 6x_2 & & & = & 4 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 4 \\ 5 & 6 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}$$

$\det(A) = 1$ $\quad$ $\det(A) \neq 0$, A is nonsingular

$$A_1(\vec{b}) = \begin{bmatrix} 3 & 2 & 3 \\ 3 & 1 & 4 \\ 4 & 6 & 0 \end{bmatrix}, \quad A_2(\vec{b}) = \begin{bmatrix} 1 & 3 & 3 \\ 0 & 3 & 4 \\ 5 & 4 & 0 \end{bmatrix}, \quad A_3(\vec{b}) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 5 & 6 & 4 \end{bmatrix}$$

$\det(A_1(\vec{b})) = 2, \quad \det(A_2(\vec{b})) = -1, \quad \det(A_3(\vec{b})) = 1$

$$x_1 = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{2}{1} = 2$$

$$x_2 = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{-1}{1} = -1$$

$$x_3 = \frac{\det(A_3(\vec{b}))}{\det(A)} = \frac{1}{1} = 1$$

$$\vec{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$$

Application

In various engineering applications, electrical or mechanical components are often chosen to try to control the long term behavior of a system (e.g. adding a damper to kill off oscillatory behavior). Using *Laplace Transforms*, differential equations are converted into algebraic equations containing a parameter s. These give rise to systems of the form

$$\begin{array}{rcrcl} 3sX & - & 2Y & = & 4 \\ -6X & + & sY & = & 1 \end{array}$$

Determine the values of s for which the system is uniquely solvable. For such s, find the solution (X, Y) using Crammer's rule.

$$\begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix} \begin{bmatrix} X \\ Y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad A = \begin{bmatrix} 3s & -2 \\ -6 & s \end{bmatrix}$$

$$\begin{aligned} 3sX &- 2Y = 4 \\ -6X &+ sY = 1 \end{aligned}$$

$\det(A) = 3s(s) - (-6)(-2)$

$= 3s^2 - 12$

$= 3(s^2 - 4)$

$\det(A) = 0$ if $s = \pm 2$

The system is uniquely solvable if $s \neq \pm 2$.

For $s \neq \pm 2$,

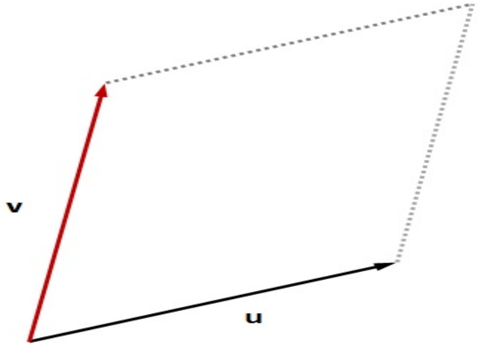
$A_1(\vec{b}) = \begin{bmatrix} 4 & -2 \\ 1 & s \end{bmatrix}$ $\det(A_1(\vec{b})) = 4s + 2$

$$A_2(\vec{b}) = \begin{bmatrix} 3s & 4 \\ -6 & 1 \end{bmatrix} \qquad \det(A_2(\vec{b})) = 3s + 24$$

$$X = \frac{\det(A_1(\vec{b}))}{\det(A)} = \frac{4s+2}{3(s^2-4)} = \frac{4s+2}{3(s-2)(s+2)}$$

$$Y = \frac{\det(A_2(\vec{b}))}{\det(A)} = \frac{3s+24}{3(s^2-4)} = \frac{s+8}{s^2-4} = \frac{s+8}{(s-2)(s+2)}$$

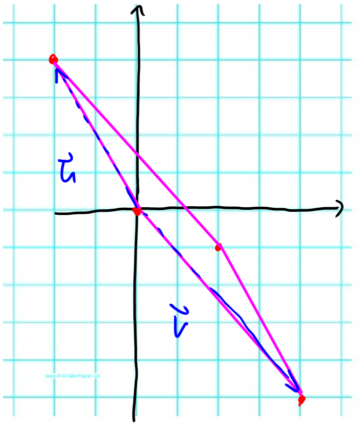
Area of a Parallelogram



Theorem: If **u** and **v** are nonzero, nonparallel vectors in $\mathbb{R}^2$, then the area of the parallelogram determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}]$.

Example

Find the area of the parallelogram with vertices $(0,0)$, $(-2,4)$, $(4,-5)$, and $(2,-1)$.



$$\vec{u} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} \qquad \vec{v} = \begin{bmatrix} 4 \\ -5 \end{bmatrix}$$

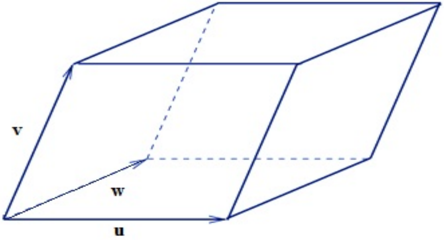
$$A = \begin{bmatrix} -2 & 4 \\ 4 & -5 \end{bmatrix}$$

$$\det(A) = -2(-5) - 4\cdot 4$$

$$= -6$$

$$\text{Area} = |\det(A)| = 6$$

Volume of a Parallelopiped



Theorem: If **u**, **v**, and **w** are nonzero, non-collinear vectors in $\mathbb{R}^3$, then the volume of the parallelopiped determined by these vectors is $|\det(A)|$ where $A = [\mathbf{u}\ \mathbf{v}\ \mathbf{w}]$.

Section 4.1: Vector Spaces and Subspaces

Definition A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Remarks

- V is more accurately called a *real vector space* when we assume that the relevant scalars are the real numbers.
- Property 1. is that V is **closed** under (a.k.a. with respect to) vector addition.
- Property 6. is that V is **closed** under scalar multiplication.
- A vector space has the same basic *structure* as $\mathbb{R}^n$
- These are **axioms.** We assume (not ”prove”) that they hold for vector space V. However, they can be used to **prove or disprove** that a given set (with operations) is actually a vector space.

Examples of Vector Spaces

For an integer $n \geq 0$, $\mathbb{P}_n$ denotes the set of all polynomials with real coefficients of degree at most n. That is

$$\mathbb{P}_n = \{\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \ldots, p_n \in \mathbb{R}\},$$

where addition[1] and scalar multiplication are defined by

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0 + q_0) + (p_1 + q_1)t + \cdots + (p_n + q_n)t^n,$$

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

[1] $\mathbf{q}(t) = q_0 + q_1 t + \cdots + q_n t^n$

Example

What is the zero vector $\mathbf{0}$ in $\mathbb{P}_n$?

Whatever $\vec{0}(t)$ is, it has to satisfy

$$(\vec{0}+\vec{p})(t) = \vec{p}(t) \quad \text{for each } \vec{p} \text{ in } \mathbb{P}_n.$$

If $\vec{0}(t) = a_0 + a_1 t + \dots + a_n t^n$

$$(\vec{0}+\vec{p})(t) = (a_0+p_0) + (a_1+p_1)t + \dots + (a_n+p_n)t^n$$

$$= p_0 + p_1 t + \dots + p_n t^n$$

$\Rightarrow a_0 = a_1 = \dots = a_n = 0$ all zero!

$$\vec{0}(t) = 0$$

Example

If $\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n$, what is the vector $-\mathbf{p}$?

Letting $-\vec{p}(t) = b_0 + b_1 t + \dots + b_n t^n$

It must be that $\vec{p} + (-\vec{p}) = \vec{0}$.

$$(\vec{p} + (-\vec{p}))(t) = \vec{p}(t) + (-\vec{p}(t))$$
$$= (p_0 + b_0) + (p_1 + b_1)t + \dots + (p_n + b_n)t^n$$
$$= 0 + 0t + \dots + 0t^n$$

$\Rightarrow b_0 = -p_0,\ b_1 = -p_1, \dots, b_n = -p_n$

$$-\vec{p}(t) = -p_0 - p_1 t - p_2 t^2 - \dots - p_n t^n$$