Mar. 20 Math 2254H sec 015H Spring 2015

Section 11.3: The Integral Test

Theorem: Let $\sum a_n$ be a series of positive terms and let the function f defined on $[1, \infty)$ be continuous, positive and decreasing with

$$a_n = f(n).$$

(i) If $\int_1^\infty f(x)\,dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(ii) If $\int_1^\infty f(x)\,dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Both series and integral converge, or both series and integral diverge.

Special Series: p-series

The series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is called a p**-series.**

Theorem: The p-series converges if $p > 1$ and diverges if $p \leq 1$.

Example: Determine if the series converges or diverges.

(a) $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^{5/2}}$

$p = \frac{5}{2} > 1$

convergent

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[3]{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$

$p = \frac{1}{3} \leq 1$

divergent.

Note About Indices

Remark: Adding or removing a **finite number of terms** from a series does NOT affect convergence!

For example: If $\sum_{n=1}^{\infty} a_n$ converges, then $\sum_{n=5}^{\infty} a_n$ converges.

If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=27}^{\infty} a_n$ also must diverge.

So we can compare sums and integrals with lower limits other than 1. That is, we can use

$$\int_5^{\infty} f(x)\,dx \quad \text{to examine the series} \quad \sum_{n=5}^{\infty} a_n$$

if f and a_n have the necessary properties.

Example

Determine if the series converges or diverges. $\displaystyle\sum_{n=3}^{\infty} \frac{n}{e^n}$

$f(x) = \dfrac{x}{e^x} = x e^{-x}$ is positive and cont. for $x \geq 3$

$f'(x) = e^{-x} - x e^{-x} = (1-x) e^{-x} < 0$ for all $x > 1$

so f is decreasing on $[3, \infty)$.

$$\int_3^{\infty} x e^{-x}\, dx = \lim_{t \to \infty} \int_3^t x e^{-x}\, dx \quad *$$

$$= \lim_{t \to \infty} \left[-(x+1) e^{-x} \right|_3^t$$

$$= \lim_{t\to\infty} \left[-(t+1)e^{-t} + (4)e^{-3} \right] *$$

$$= 0 + 4e^{-3}$$

The integral converges. The series converges by the Integral test.

$$* \int x e^{-x}\,dx \qquad u = x \qquad du = dx$$

$$v = -e^{-x} \qquad dv = e^{-x}\,dx$$

$$= -xe^{-x} - \int (-e^{-x})\,dx$$

$$= -xe^{-x} + \int e^{-x}\,dx$$

$$= -xe^{-x} - e^{-x} + C$$

$$* \quad \lim_{t\to\infty} (t+1)e^{-t} = \text{"}\infty \cdot 0\text{"}$$

use

$$= \lim_{t\to\infty} \frac{t+1}{e^t} = \text{"}\frac{\infty}{\infty}\text{"}$$ l'H rule

$$= \lim_{t\to\infty} \frac{1}{e^t} = 0$$