March 20 Math 2306 sec. 57 Spring 2018

Section 12: LRC Series Circuits

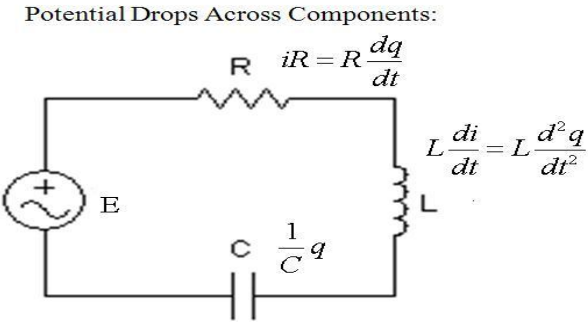

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

This is a second order, linear, constant coefficient nonhomogeneous (if $E \neq 0$) equation.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

$$\begin{array}{ll} \textbf{overdamped if} & R^2 - 4L/C > 0, \\ \textbf{critically damped if} & R^2 - 4L/C = 0, \\ \textbf{underdamped if} & R^2 - 4L/C < 0. \end{array}$$

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

Steady and Transient States

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$$q(t) = q_c(t) + q_p(t).$$

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

We found that the ODE in standard form is

$$q'' + 20q' + 500q = 10\cos(10t)$$

and the complementary solution is
$q_c = c_1 e^{-10t}\cos(20t) + c_2 e^{-10t}\sin(20t)$.

Using undetermined coefficients

$$q_p = A\cos(10t) + B\sin(10t)$$

$$q_p' = -10A\sin(10t) + 10B\cos(10t)$$

$$y_p'' = -100A\cos(10t) - 100B\sin(10t)$$

$$y_p'' + 20y_p' + 500y_p =$$

$$-100A\cos(10t) - 100B\sin(10t) + 20\left(-10A\sin(10t) + 10B\cos(10t)\right) + 500\left(A\cos(10t) + B\sin(10t)\right)$$

$$= 10\cos(10t) + 0\sin(10t)$$

$$\cos(10t)\left(-100A + 200B + 500A\right) + \sin(10t)\left(-100B - 200A + 500B\right)$$

$$= 10\cos(10t) + 0\sin(10t)$$

$$400A + 200B = 10$$

$$-200A + 400B = 0 \Rightarrow 400B = 200A$$

$$2B = A$$

$$800B + 200B = 10$$

$$1000B = 10 \Rightarrow B = \frac{10}{1000} = \frac{1}{100}$$

$$A = 2B = \frac{2}{100} = \frac{1}{50}$$

So the steady state charge

$$q_p = \frac{1}{50}\cos(10t) + \frac{1}{100}\sin(10t)$$

Current $i = \frac{dq}{dt}$, so the steady state

current

$$i_p = \frac{dq_p}{dt} = \frac{-10}{50}\sin(10t) + \frac{10}{100}\cos(10t)$$

$$\boxed{i_p = \frac{1}{10}\cos(10t) - \frac{1}{5}\sin(10t)}$$

Section 13: The Laplace Transform

A quick word about functions of 2-variables:

Suppose $G(s, t)$ is a function of two independent variables (s and t) defined over some rectangle in the plane $a \le t \le b$, $c \le s \le d$. If we compute an integral with respect to one of these variables, say t,

$$\int_{\alpha}^{\beta} G(s, t)\, dt$$

- the result is a function of the remaining variable s, and
- the variable s is treated as a constant while integrating with respect to t.

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t) + \beta g(t))\,dt = \alpha \int_a^b K(s,t)f(t)\,dt + \beta \int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.

Note 2: If we take s to be real-valued, then

$$\lim_{t\to\infty} e^{-st} = 0 \quad \text{if } s > 0, \quad \text{and} \quad \lim_{t\to\infty} e^{-st} = \infty \quad \text{if } s < 0.$$

Find the Laplace transform of $f(t) = 1$

By definition

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st} \cdot 1 \, dt$$

Case 1 : $s = 0$ the integral is $\int_0^\infty dt$ which is divergent

zero is not in the domain.

Case 2 : $s \neq 0$

$$\int_0^\infty e^{-st} dt = \lim_{b \to \infty} \int_0^b e^{-st} dt = \lim_{b \to \infty} \frac{-1}{s} e^{-st} \Big|_0^b$$

$$= \lim_{b \to \infty} \frac{-1}{s} \left(e^{-sb} - e^{0} \right)$$

only convergent if $s > 0$

$$= \frac{-1}{s}(0-1) = \frac{1}{s} \quad \text{for } s>0$$

So $\mathcal{L}\{1\} = \frac{1}{s}$ with domain $s>0$.

Find the Laplace transform of $f(t) = t$

By definition

$$\mathcal{L}\{t\} = \int_0^\infty e^{-st}\, t\, dt$$

Case 1: $s=0$ the integral is $\int_0^\infty t\, dt$ which diverges.

zero is not in the domain.

Case 2: $s \neq 0$

parts

$u = t \qquad du = dt$

$v = \frac{-1}{s} e^{-st} \qquad dv = e^{-st}\, dt$

$$\int_0^\infty e^{-st}\, t\, dt$$

$$= \frac{-1}{s} t e^{-st} \Big|_0^\infty - \int_0^\infty \frac{-1}{s} e^{-st}\, dt$$

we require $s > 0$ for convergence

$$= \frac{-1}{s}(0-0) + \frac{1}{s}\int_0^\infty e^{-st}\,dt$$

$*$ If $s>0$

$$\lim_{t\to\infty} te^{-st} = 0$$

$$= \frac{1}{s}\underbrace{\int_0^\infty e^{-st}\,dt}_{=\,\mathcal{L}\{1\}} \quad \text{for } s>0.$$

$$= \frac{1}{s}\left(\frac{1}{s}\right) = \frac{1}{s^2}$$

So $\mathcal{L}\{t\} = \frac{1}{s^2}$ for $s>0$

A piecewise defined function

Find the Laplace transform of f defined by

$$f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$$

By definition $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt$

$$\mathcal{L}\{f(t)\} = \int_0^{10} e^{-st} f(t)\,dt + \int_{10}^\infty e^{-st} f(t)\,dt$$

$$= \int_0^{10} e^{-st} (2t)\,dt + \int_{10}^\infty e^{-st} (0)\,dt$$

For $s=0$, the transform is

$$\int_0^{10} 2t\,dt = t^2 \Big|_0^{10} = 100$$

For $s \neq 0$

$$\mathcal{L}\{f(t)\} = \int_0^{10} 2e^{-st}\,t\,dt$$

$$= 2\left(\frac{-1}{s}te^{-st}\Big|_0^{10} - \frac{1}{s^2}e^{-st}\Big|_0^{10}\right)$$

$$= 2\left(\frac{-1}{s}10e^{-10s} - 0 - \frac{1}{s^2}\left(e^{-10s} - e^{0}\right)\right)$$

$$= \frac{-20}{s}e^{-10s} - \frac{2}{s^2}e^{-10s} + \frac{2}{s^2}$$

So
$$\mathcal{L}\{f(t)\} = \begin{cases} \dfrac{2}{s^2} - \dfrac{2}{s^2}e^{-10s} - \dfrac{20}{s}e^{-10s}, & s \neq 0 \\ 100, & s = 0 \end{cases}$$

Left as an exercise: Show that this is continuous @ $s=0$.

The Laplace Transform is a Linear Transformation

Some basic results include:

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$

- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$

- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$

- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$

- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$

- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Examples: Evaluate

(a) $f(t) = \cos(\pi t)$

well use

$$\mathscr{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}, \quad s>0$$

$$\mathscr{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}, \quad s>0$$

Examples: Evaluate

Use $\mathcal{L}\{1\} = \frac{1}{s}$ $\quad s>0$

$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$ $\quad s>0$

$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ $\quad s>a$

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{2t^4 - e^{-5t} + 3\}$$

$$2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$2\left(\frac{4!}{s^{4+1}}\right) - \frac{1}{s-(-5)} + 3\left(\frac{1}{s}\right) = \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s}$$

$s>0 \qquad s>-5 \qquad s>0$

for $s>0$