March 20 Math 3260 sec. 55 Spring 2018

Section 4.3: Linearly Independent Sets and Bases

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

Standard Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$. For example, the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{respectively.}$$

Other Vector Spaces

Other vector spaces have **standard** bases as well. For example,

$$\{1, t, t^2, \ldots, t^n\} \quad \text{is a standard basis for } \mathbb{P}_n$$

$$\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\} \text{ is a standard basis for } M^{2\times 2}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Theorem:

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space V and $H =$Span(S).

(a.) If one of the vectors in S, say $\mathbf{v}_k$ is a linear combination of the other vectors in S, then the subset of S obtained by eliminating $\mathbf{v}_k$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis for H.

If we start with a spanning set, we can eliminate *duplication* and arrive at a basis.

Column Space

Find a basis for the column space matrix B that is in reduced row echelon form

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$= [\vec{b}_1\ \vec{b}_2\ \vec{b}_3\ \vec{b}_4\ \vec{b}_5]$

Col B is the subspace of $\mathbb{R}^4$ spanned by the columns of B. To get a basis, we need a linearly independent subset of B's columns.

By observation:

$\vec{b}_2 = 4\vec{b}_1$, $\vec{b}_3$ doesn't depend on $\vec{b}_1$

toss $\vec{b}_2$ keep $\vec{b}_1$, $\vec{b}_3$

$\vec{b}_4 = 2\vec{b}_1 - \vec{b}_3$

toss $\vec{b}_4$

$\vec{b}_5$ doesn't depend on $\vec{b}_1$ or $\vec{b}_3$ or $\{\vec{b}_1, \vec{b}_3\}$

keep $\vec{b}_5$

A basis for Col B is

$$\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$$

Using the rref

Theorem: If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ are row equivalent matrices, then Nul $A =$ Nul B. That is, the equations

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set.

Note what this means! It means that $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ have **exactly the same linear dependence relationships**!

Theorem:

The pivot columns of a matrix A form a basis of Col A.

Caveat: This means we can use row reduction to identify a basis, but the vectors we obtain will be from the original matrix A. (As illustrated in the following example.)

Find a basis for Col A[1]

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

we'll use the rref to identify pivot columns.

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

pivot columns are 1, 3, and 5.

we use the 1st, 3rd, 5th columns of $\underline{\underline{A}}$.

$$\text{Col } A = \text{Span}\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

This set is a basis.

Find bases for Nul A and Col A

$$A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 5 & 1 \end{bmatrix}$$

We can get both from the rref.

For Col A we identify pivot columns.

For Nul A, we use it to find the solution set of $A\vec{x} = \vec{0}$.

$$A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 5 \end{bmatrix}$$

1st two columns are pivot columns.

A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

For Nul A, note $A\vec{x}=\vec{0}$ $\Rightarrow$

$$x_1 = -3x_3 + 2x_4$$
$$x_2 = x_3 - 5x_4$$
$$x_3, x_4 - \text{free}$$

For $\vec{x}$ in Nul A

$$\vec{x} = x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

A basis for Nul A is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$.

Section 4.4: Coordinate Systems

We begin with a theorem about uniqueness of linear combinations (of linearly independent vectors).

Theorem: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Then for each vector $\mathbf{x}$ in V, there is a unique set of scalars $c_1, \ldots, c_n$ such that

$$\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n.$$

To see this, suppose

$$\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_n\vec{b}_n \quad \text{and}$$

$$\vec{x} = a_1\vec{b}_1 + a_2\vec{b}_2 + \cdots + a_n\vec{b}_n$$

subtract the 2^{nd} line from the first

$$\vec{0} = (c_1 - a_1)\vec{b}_1 + (c_2 - a_2)\vec{b}_2 + \cdots + (c_n - a_n)\vec{b}_n$$

$\mathcal{B}$ is a basis! The $\vec{b}_i$ vectors are linearly independent. Hence

$$\left.\begin{array}{l} c_1 - a_1 = 0 \\ c_2 - a_2 = 0 \\ \vdots \\ c_n - a_n = 0 \end{array}\right\} \implies \begin{array}{l} a_1 = c_1 \\ a_2 = c_2 \\ \vdots \\ a_n = c_n \end{array}$$

The coefficients are unique for a given $\vec{x}$.

Coordinate Vectors

Definition: Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ where these entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

Example

Let $\mathcal{B} = \{1, t, t^2, t^3\}$ (in that order) in $\mathbb{P}_3$. Determine $[\mathbf{p}]_{\mathcal{B}}$ where

(a) $\mathbf{p}(t) = 3 - 4t^2 + 6t^3$ $= (3)1 + (0)t + (-4)t^2 + 6t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 0 \\ -4 \\ 6 \end{bmatrix}$$

(b) $\mathbf{p}(t) = p_0 + p_1 t + p_2 t^2 + p_3 t^3$

$$[\vec{p}]_{\mathcal{B}} = \begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ p_3 \end{bmatrix}$$

Example

Let $\mathbf{b}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Find $[\mathbf{x}]_{\mathcal{B}}$ for $\mathbf{x} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$.

$[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ where $\vec{x} = c_1\vec{b}_1 + c_2\vec{b}_2$

So

$$c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

in matrix format

$$\begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix}$$

Let's use a matrix inverse to solve (not the only option!).

If $A = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}$, $\det(A) = 3$ so $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$

So $[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ 5 \end{bmatrix}$

$= \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$

Coordinates in $\mathbb{R}^n$

Note from this example that $\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ where $P_{\mathcal{B}}$ is the matrix $[\mathbf{b}_1\ \mathbf{b}_2]$. The matrix $P_{\mathcal{B}}$ is called the **change of coordinates matrix** for the basis $\mathcal{B}$ (or from the basis $\mathcal{B}$ to the standard basis).

Let $\mathcal{B} = \{\mathbf{b}_1, \dots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the linear transformation defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

where the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$