March 21 Math 1190 sec. 62 Spring 2017

Section 4.2: Maximum and Minimum Values; Critical Numbers

Let's start with some review questions inspired by last Thursday's quiz!

Question

Let $y = x^{e^x}$. Which of the following is true?

(a) The power rule gives $y' = (e^x)x^{e^x-1}$.

(b) The exponential rule gives $y' = x^{e^x} \ln x$.

(c) y is the product of x and e^x.

(d) There is no derivative rule that applies directly to y.

power function

x^n ← constant power

↑ variable base

exponential function

a^x ← variable power

↑ constant base

Question

Let $y = x^{e^x}$. Which of the following is true?

(a) $y = e^x \ln x$

(b) $\ln y = e^x \ln x$

(c) $\frac{1}{y} = e^x \frac{1}{x}$

(d) $\ln y = e \ln x^x$

$$\ln y = \ln x^{e^x}$$
$$= e^x \ln x$$

Question

Given that $\ln y = e^x \ln x$,

(a) $\dfrac{dy}{dx} = e^x \ln x + \dfrac{e^x}{x}$

(b) $\dfrac{dy}{dx} = x e^x \left(\dfrac{e^x}{x} \right)$

(c) $\dfrac{dy}{dx} = \dfrac{e^x}{x}$

(d) $\dfrac{dy}{dx} = x^{e^x} \left(e^x \ln x + \dfrac{e^x}{x} \right)$

implicit

product rule

$\dfrac{d}{dx} \ln y = \dfrac{d}{dx} e^x \ln x$

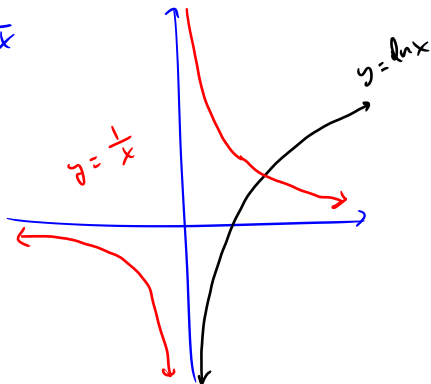$\dfrac{1}{y} \dfrac{dy}{dx} = e^x \ln x + e^x \cdot \dfrac{1}{x}$

$\dfrac{dy}{dx} = y \left(e^x \ln x + \dfrac{e^x}{x} \right)$

$= x^{e^x} \left(e^x \ln x + \dfrac{e^x}{x} \right)$

Question

True or False: $\ln x = \frac{1}{x}$

But $\quad \frac{d}{dx} \ln x = \frac{1}{x}$

$y = \frac{1}{x}$

$y = \ln x$

Question

Given $y = \ln(\ln x)$, find y'.

(a) $y' = \dfrac{1}{x \ln x}$

(b) $y' = \dfrac{1}{\ln(\ln x)}$

(c) $y' = \dfrac{1}{(\ln x)^2}$

(d) $y' = \dfrac{1}{\ln x}$

If $\quad u = \ln x \quad$ and $\quad f(u) = \ln u$

$$y = f(u)$$

$$y' = f'(u) \cdot u'(x)$$

$$= \frac{1}{u} \cdot \frac{1}{x} = \frac{1}{x \ln x}$$

$$y' = \frac{\frac{1}{x}}{\ln x} = \frac{1}{x \ln x}$$

Section 4.3: The Mean Value Theorem

Recall

Rolle's Theorem: Let f be a function that is

  i continuous on the closed interval $[a, b]$,

  ii differentiable on the open interval (a, b), and

  iii such that $f(a) = f(b)$.

Then there exists a number c in (a, b) such that $f'(c) = 0$.

If f takes the same y-value at both ends, and is continuous and *smooth* in between them, then it's either horizontal, or it turns around. Somewhere there must be a horizontal tangent!

The Mean Value Theorem

Theorem: Suppose f is a function that satisfies

  i f is continuous on the closed interval $[a, b]$, and
  ii f is differentiable on the open interval (a, b).

Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

If f is continuous and *smooth* between the point $(a, f(a))$ and $(b, f(b))$, then somewhere between them is a point where the tangent is parallel to the secant line connecting the ends.

Rate of Change and Slope

From day 1 of class: Given a line $y = mx + b$, and a change in the independent variable Δx,

the change in the dependent variable is

$$\Delta y = m\Delta x.$$

Now look at the conclusion of the MVT!

$$f(b) - f(a) \quad = \quad f'(c) \qquad (b-a)$$

$$\text{change in y} \quad = \quad \text{a slope} \quad \text{change in x}$$

$$\Delta y \qquad = \quad m \qquad\qquad \Delta x$$

A couple of questions we can ask...

An Interesting Question: We know that $f'(x) = 0$ if f is a constant function. Is the converse true? That is, if $f'(x) = 0$ does that mean that f is a constant function?

Another Interesting Question: If we have two functions f and g such that on some interval

$$f'(x) = g'(x),$$

does this mean that f and g have to be the same function?

An Interesting Question: We know that $f'(x) = 0$ if f is a constant function. Is the converse true? That is, if $f'(x) = 0$ does that mean that f is a constant function?

yes !

Theorem: If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

only constant functions have derivatives
that are zero on an interval.

Important Consequence of the MVT

Another Interesting Question: If we have two functions f and g such that on some interval

$$f'(x) = g'(x),$$

does this mean that f and g have to be the same function? No

But they can only differ by an added constant.

Corollary: If $f'(x) = g'(x)$ for all x in an interval (a, b), then $f - g$ is constant on (a, b). In other words,

$$f(x) = g(x) + C \quad \text{where } C \text{ is some constant.}$$

Examples

Find all possible functions $f(x)$ that satisfy the condition

(a) $f'(x) = \cos x$ on $(-\infty, \infty)$

Since $\frac{d}{dx} \sin x = \cos x$

all functions can only differ by an added constant. So

$f(x) = \sin x + C$ for any constant C.

(b) $f'(x) = 2x$ on $(-\infty, \infty)$

$$\frac{d}{dx} x^2 = 2x$$

So all such functions are

$$f(x) = x^2 + C \quad \text{for arbitrary constant } C.$$

Question

Find all possible functions $h(t)$ that satisfy the condition

(c) $\quad h'(t) = \sec^2 t \quad$ on $\quad \left(-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right)$

(a) $h(t) = \sec^2 t + C$

(b) $h(t) = \tan t + 1$

(c) $h(t) = \tan t + C$

Another Consequence of the MVT

Another significant consequence of the MVT is that it provides a test for the increasing and decreasing behavior of a differentiable function.

Theorem: Let f be differentiable on an open interval (a, b). If

- $f'(x) > 0$ on (a, b), the f is increasing on (a, b), and if
- $f'(x) < 0$ on (a, b), the f is decreasing on (a, b).

Example

Determine the intervals over which *f* is increasing and the intervals over which it is decreasing where

$$f(x) = 6x^4 - 32x^3 - 7 \qquad \text{Domain is } (-\infty, \infty)$$

- find f'
- find where $f'(x) = 0$ or where $f'(x)$ is undefined.
- Consider the intervals determined by the numbers found.
- Do a sign analysis by checking the sign of f' in each interval.
- State our conclusion

$$f(x) = 6x^4 - 32x^3 - 7$$

$$f'(x) = 24x^3 - 96x^2 = 24x^2(x-4)$$

f' is always defined, $f'(x) = 0 \Rightarrow$

$$24x^2(x-4) = 0$$

$$\Rightarrow \quad x^2 = 0 \quad \text{or} \quad x - 4 = 0$$

$$x = 0 \quad \text{or} \quad x = 4$$

This breaks the domain into 3 intervals,

$$(-\infty, 0), \ (0, 4), \ \text{and} \ (4, \infty).$$

$$f'(x) = 24x^2(x-4)$$

Sign analysis

Pick test points in each interval

test   -1 $f'(-1) = 24(-1)^2(-1-4) = -120$ negative

+1 $f'(1) = 24(1)^2(1-4) = -72$ negative

5      $f'(5) = 24(5)^2(5-4) = 600$ positive.

Based on the analysis

 f is decreasing on $(-\infty, 0) \cup (0,4)$.

 f is increasing on $(4, \infty)$.

* The conclusion is about the behavior of
  f on intervals.

Question

Suppose that we compute the derivative of some function *g* and find

$$g'(x) = (2 + x)e^{x/2}.$$

Determine the intervals over which *g* is increasing and over which it is decreasing.

(a) *g* is increasing on $(-1/2, \infty)$ and decreasing on $(-\infty, -1/2)$.

(b) *g* is increasing on $(-2, \infty)$ and decreasing on $(-\infty, -2)$.

(c) *g* is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

(d) *g* is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$.

Section 4.4: Local Extrema and Concavity

We have already seen that the first derivative f' can tell us about the behaviour of the function f—in particular, it gives information about where it is increasing or decreasing, and where it may take a local extreme value.

In this section, we'll expand on that as well as introduce information about a function that can be deduced from the nature of its second derivative.

Theorem: First derivative test for local extrema

Let f be continuous and suppose that c is a critical number of f.

- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' does not change signs at c, then f does not have a local extremum at c.

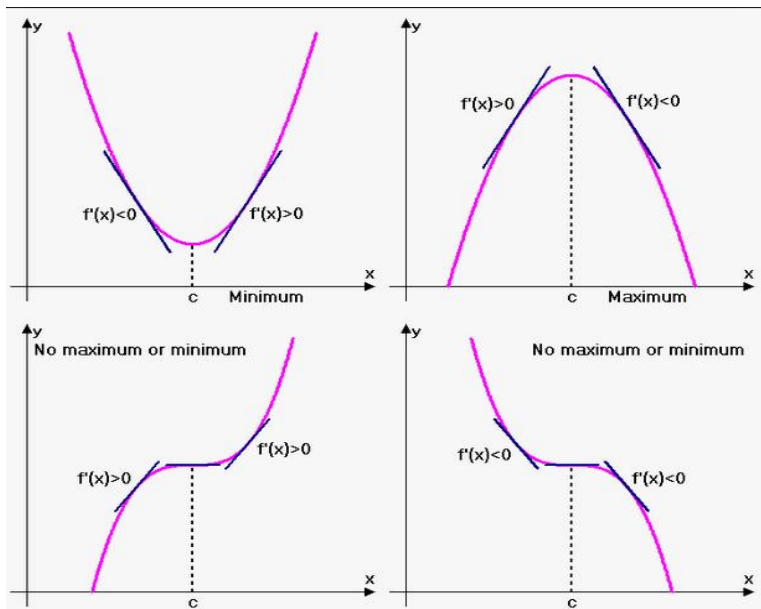Note: we read from left to right as usual when looking for a sign change.

Figure: First derivative test

Example

Find all the critical points of the function and classify each one as a local maximum, a local minimum, or neither.

$$f(x) = 16\sqrt[3]{x} - x\sqrt[3]{x} \; = 16\,x^{1/3} - x^{4/3}$$

first we'll do the same sign analysis.

$$f'(x) = 16\left(\frac{1}{3}x^{-2/3}\right) - \frac{4}{3}x^{1/3}$$

$$= \frac{16}{3}x^{-2/3} - \frac{4}{3}x^{1/3} \; = \frac{4}{3}x^{-2/3}(4-x)$$

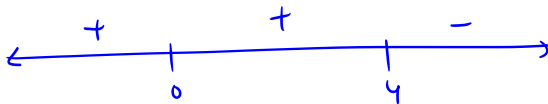$$= \frac{4(4-x)}{3x^{2/3}}$$

$f'(x) = 0$ if the numerator is zero

$$4(4-x) = 0 \implies x = 4$$

$f'(x)$ is undefined if the denominator is zero

$$3x^{2/3} = 0 \implies x = 0$$

The domain is all reals, and we have two critical numbers.

$$f'(x) = \frac{4(4-x)}{3x^{2/3}}$$

Test points:

$$-1 \qquad f'(-1) = \frac{4(4-(-1))}{3(-1)^{2/3}} \qquad \frac{(+)}{(+)} \qquad \text{positive}$$

$$1 \qquad f'(1) = \frac{4(4-1)}{3(1)^{2/3}} \qquad \frac{(+)}{(+)} \qquad \text{positive}$$

$$5 \qquad f'(5) = \frac{4(4-5)}{3(5)^{2/3}} \qquad \frac{(-)}{(+)}$$

Critical number zero is neither a local
max nor min.

Critical number 4 corresponds to
a local max.

Question

Consider the function $f(t) = t^4 + 4t^3$. Which of the following is true about this function?

(a) f has no critical numbers.

(b) f has critical numbers 0 and -4.

(c) f has critical numbers 4 and 12.

(d) f has critical numbers 0 and -3.

$f'(t) = 4t^3 + 12t^2$

$= 4t^2(t+3)$

Question

Consider the function $f(t) = t^4 + 4t^3$. Which of the following is true about this function?

(a) f has a local minimum at $t = 0$ and a local maximum at $t = -3$.

(b) f has a local minimum at $t = -3$ and a local maximum at $t = 0$.

(c) f has a local minimum at $t = -3$.

(d) f has a local minimum at $t = 0$.

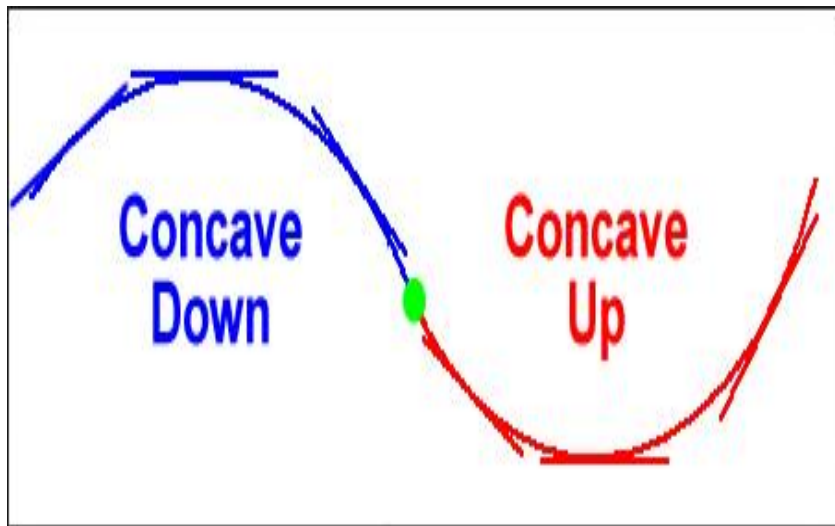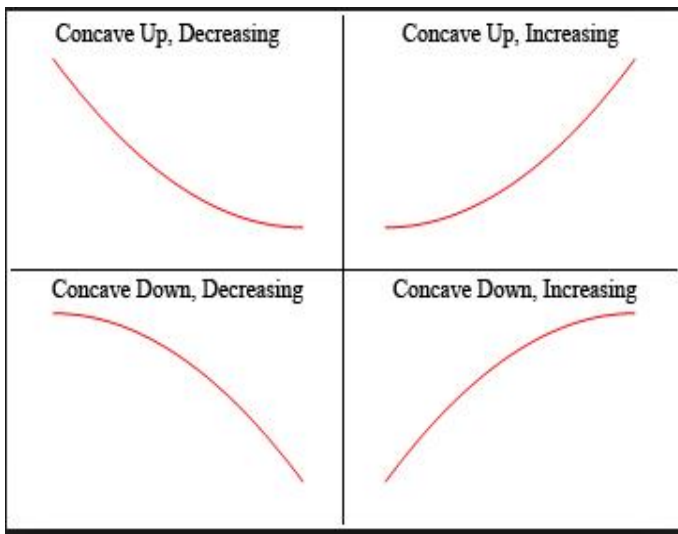$$f'(t) = 4t^2(t+3)$$

Concavity and The Second Derivative

Concavity: refers to the *bending* nature of a graph. In particular, a curve is concave down if it's cupped side is down, and it is concave up if it's cupped upward.

Concavity
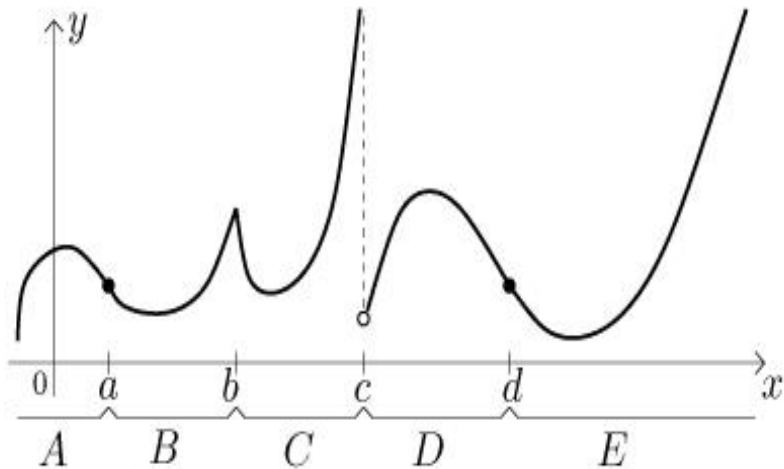

Figure

Figure: A graph can have either increasing or decreasing behavior and be either concave up or down.

Figure: We can consider concavity at a point, but it's best thought of as a property over an interval. Many function's graphs have concavity that changes over the domain.

Definition of Concavity

Concave Up: If the graph of a function *f* lies above all of its tangent lines over an interval *I*, then *f* is concave up on *I*.

Concave Down: If the graph of *f* lies below each of its tangent lines on an interval *I*, *f* is concave down on *I*.

Theorem: (Second Derivative Test for Concavity)
Suppose *f* is twice differentiable on an interval *I*.

▶ If $f''(x) > 0$ on *I*, then the graph of *f* is concave up on *I*.

▶ If $f''(x) < 0$ on *I*, then the graph of *f* is concave down on *I*.

Definition: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous at P and the concavity of f changes at P (from down to up or from up to down). A point where $f''(x) = 0$ would be a candidate for being an inflection point.