March 21 Math 1190 sec. 63 Spring 2017

Section 4.2: Maximum and Minimum Values; Critical Numbers

Let's start with some review questions inspired by last Thursday's quiz!

Question

Let $y = x^{x^2}$. Which of the following is true?

(a) The power rule gives $y' = (x^2)x^{x^2-1}$.

(b) The exponential rule gives $y' = x^{x^2}\ln x$.

(c) y is the product of x and x^2.

(d) There is no derivative rule that applies directly to y.

Power function

x^n ← constant power

variable base

Exponential function

a^x ← variable power

constant base

Question

Let $y = x^{x^2}$. Which of the following is true?

(a) $y = x^2 \ln x$

(b) $\ln y = x^2 \ln x$

(c) $\frac{1}{y} = x^2 \frac{1}{x}$

(d) $\ln y = 2x \ln x$

$\underbrace{\ln y}_{\text{chain}} = \ln x^{x^2} = \underbrace{x^2 \ln x}_{\text{product}}$

Question

implicit product

Given that $\ln y = x^2 \ln x$,

(a) $\dfrac{dy}{dx} = 2x \ln x + x$

(b) $\dfrac{dy}{dx} = 2x^{x^2}$

(c) $\dfrac{dy}{dx} = 2$

(d) $\dfrac{dy}{dx} = x^{x^2}(2x \ln x + x)$

$$\frac{d}{dx} \ln y = \frac{d}{dx} x^2 \ln x$$

$$\frac{1}{y}\frac{dy}{dx} = 2x \ln x + x^2 \frac{1}{x}$$

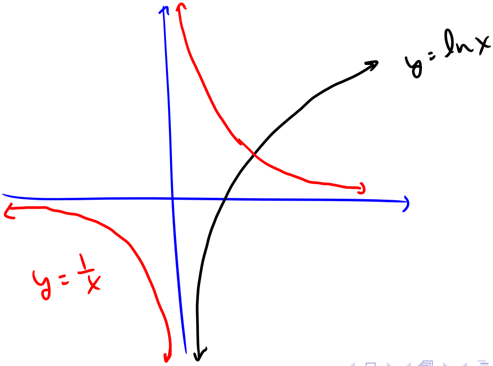
$$\frac{dy}{dx} = y(2x \ln x + x)$$

$$= x^{x^2}(2x \ln x + x)$$

Question

True or False: $\ln x = \frac{1}{x}$

it is true that $\frac{d}{dx} \ln x = \frac{1}{x}$



Question

Given $y = \ln(\ln x)$, find y'.

$$\frac{dy}{dx} = \frac{\frac{d}{dx}\ln x}{\ln x} = \frac{\frac{1}{x}}{\ln x}$$

$$= \frac{1}{x\ln x}$$

(a) $y' = \dfrac{1}{x \ln x}$

(b) $y' = \dfrac{1}{\ln(\ln x)}$

(c) $y' = \dfrac{1}{(\ln x)^2}$

(d) $y' = \dfrac{1}{\ln x}$

Section 4.3: The Mean Value Theorem

Recall

Rolle's Theorem: Let f be a function that is

i continuous on the closed interval $[a, b]$,

ii differentiable on the open interval (a, b), and

iii such that $f(a) = f(b)$.

Then there exists a number c in (a, b) such that $f'(c) = 0$.

If f takes the same y-value at both ends, and is continuous and *smooth* in between them, then it's either horizontal, or it turns around. Somewhere there must be a horizontal tangent!

The Mean Value Theorem

Theorem: Suppose f is a function that satisfies

i f is continuous on the closed interval $[a, b]$, and
ii f is differentiable on the open interval (a, b).

Then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \quad \text{equivalently} \quad f(b) - f(a) = f'(c)(b - a).$$

If f is continuous and *smooth* between the point $(a, f(a))$ and $(b, f(b))$, then somewhere between them is a point where the tangent is parallel to the secant line connecting the ends.

Rate of Change and Slope

From day 1 of class: Given a line $y = mx + b$, and a change in the independent variable Δx,

the change in the dependent variable is

$$\Delta y = m\Delta x.$$

Now look at the conclusion of the MVT!

$$\begin{array}{ccccc} f(b)-f(a) & = & f'(c) & (b-a) \\ \text{change in y} & = & \text{a slope} & \text{change in x} \\ \Delta y & = & m & \Delta x \end{array}$$

A couple of questions we can ask...

An Interesting Question: We know that $f'(x) = 0$ if f is a constant function. Is the converse true? That is, if $f'(x) = 0$ does that mean that f is a constant function?

Another Interesting Question: If we have two functions f and g such that on some interval

$$f'(x) = g'(x),$$

does this mean that f and g have to be the same function?

Important Consequence of the MVT

An Interesting Question: We know that $f'(x) = 0$ if f is a constant function. Is the converse true? That is, if $f'(x) = 0$ does that mean that f is a constant function?

Yes.

Theorem: If $f'(x) = 0$ for all x in an interval (a, b), then f is constant on (a, b).

The only function with derivative zero on an interval is a constant function.

Important Consequence of the MVT

Another Interesting Question: If we have two functions f and g such that on some interval

$$f'(x) = g'(x),$$

does this mean that f and g have to be the same function? No.

They can differ, but at most by an added constant.

Corollary: If $f'(x) = g'(x)$ for all x in an interval (a,b), then $f - g$ is constant on (a,b). In other words,

$$f(x) = g(x) + C \quad \text{where } C \text{ is some constant.}$$

Examples

Find all possible functions $f(x)$ that satisfy the condition

(a) $f'(x) = \cos x$ on $(-\infty, \infty)$

$$\frac{d}{dx} \operatorname{Sin} x = \operatorname{Cos} x$$

So all such functions are of the form

$f(x) = \operatorname{Sin} x + C$ for arbitrary constant C.

(b) $f'(x) = 2x$ on $(-\infty, \infty)$

$$\frac{d}{dx} x^2 = 2x$$

so all functions look like

$f(x) = x^2 + C$ for abitrary constant C.

Question

Find all possible functions $h(t)$ that satisfy the condition

$$\text{(c)} \quad h'(t) = \sec^2 t \quad \text{on} \quad \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$

(a) $h(t) = \sec^2 t + C$

(b) $h(t) = \tan t + 1$

(c) $h(t) = \tan t + C$

Another Consequence of the MVT

Another significant consequence of the MVT is that it provides a test for the increasing and decreasing behavior of a differentiable function.

Theorem: Let f be differentiable on an open interval (a, b). If

- $f'(x) > 0$ on (a, b), the f is increasing on (a, b), and if
- $f'(x) < 0$ on (a, b), the f is decreasing on (a, b).

Example

Determine the intervals over which f is increasing and the intervals over which it is decreasing where

$$f(x) = 6x^4 - 32x^3 - 7$$

- find $f'(x)$
- Determine where $f'(x)=0$ and where $f'(x)$ is undefined.
- we split the domain into intervals determined by these numbers.
- test the sign of f' in each interval
- base a conclusion on the sign analysis.

The domain for $f(x) = 6x^4 - 32x^3 - 7$ is $(-\infty, \infty)$.

$$f'(x) = 24x^3 - 96x^2 = 24x^2(x-4)$$

$f'(x)$ is never undefined

$$f'(x) = 0 \implies 24x^2(x-4) = 0$$

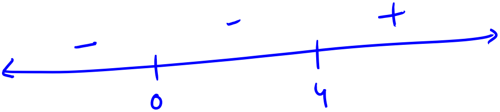
$$\implies x^2 = 0 \quad \text{or} \quad x - 4 = 0$$

$$x = 0 \quad \text{or} \quad x = 4$$

These divide the domain into 3 intervals

$(-\infty, 0)$, $(0, 4)$, and $(4, \infty)$.

$$f'(x) = 24x^2(x-4)$$



sign of f'

Test the sign:

test pt -1 , $f'(-1) = 24(-1)^2(-1-4) = -120$ negative

1 , $f'(1) = 24(1)^2(1-4) = -72$ negative

5 , $f'(5) = 24(5)^2(5-4) = 600$ positive

f is decreasing on $(-\infty, 0) \cup (0, 4)$

f is increasing on $(4, \infty)$.

* Note that the conclusion is over intervals.

Question

Suppose that we compute the derivative of some function g and find

$$g'(x) = (2 + x)e^{x/2}.$$

$-$ $+$

-2

Determine the intervals over which g is increasing and over which it is decreasing.

(a) g is increasing on $(-1/2, \infty)$ and decreasing on $(-\infty, -1/2)$.

(b) g is increasing on $(-2, \infty)$ and decreasing on $(-\infty, -2)$.

(c) g is increasing on $(2, \infty)$ and decreasing on $(-\infty, 2)$.

(d) g is increasing on $(-\infty, -2)$ and decreasing on $(-2, \infty)$.

Section 4.4: Local Extrema and Concavity

We have already seen that the first derivative f' can tell us about the behaviour of the function f—in particular, it gives information about where it is increasing or decreasing, and where it may take a local extreme value.

In this section, we'll expand on that as well as introduce information about a function that can be deduced from the nature of its second derivative.

Theorem: First derivative test for local extrema

Let f be continuous and suppose that c is a critical number of f.

- If f' changes from negative to positive at c, then f has a local minimum at c.
- If f' changes from positive to negative at c, then f has a local maximum at c.
- If f' does not change signs at c, then f does not have a local extremum at c.

Note: we read from left to right as usual when looking for a sign change.

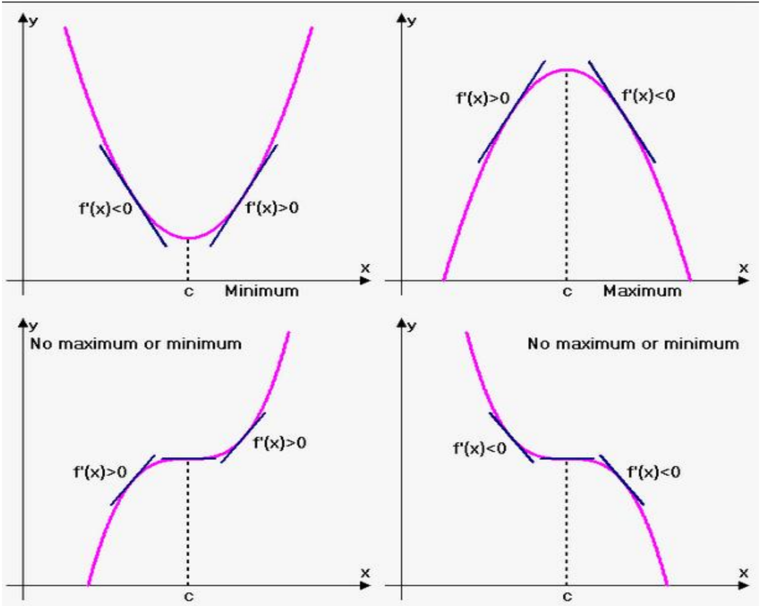


Figure: First derivative test

Example

Find all the critical points of the function and classify each one as a local maximum, a local minimum, or neither.

$$f(x) = 16\sqrt[3]{x} - x\sqrt[3]{x} = 16x^{1/3} - x^{4/3}$$

- We do the same sign analysis as before, then
- classify critical numbers based on the results.

$$f'(x) = 16\left(\frac{1}{3}x^{-2/3}\right) - \frac{4}{3}x^{1/3}$$

$$= \frac{16}{3}x^{-2/3} - \frac{4}{3}x^{1/3}$$

$$= \frac{4}{3}x^{-2/3}(4-x) = \frac{4(4-x)}{3x^{2/3}}$$

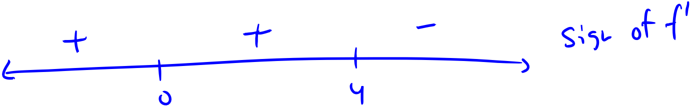
The domain of f is $(-\infty, \infty)$.

Find critical #'s.

$f'(x) = 0$ if $4(4-x) = 0 \implies x = 4$

$f'(x)$ is undefined if $3x^{2/3} = 0 \implies x = 0$.

$$f'(x) = \frac{4(4-x)}{3x^{2/3}}$$



Test pts: -1 $\quad f'(-1) = \dfrac{4(4-(-1))}{3(-1)^{2/3}} \quad \dfrac{(+)}{(+)}$ positive

$1 \qquad f'(1) = \dfrac{4(4-1)}{3(1)^{2/3}} \qquad \dfrac{(+)}{(+)}$ positive

$5 \qquad f'(5) = \dfrac{4(4-5)}{3(5)^{2/3}} \qquad \dfrac{(-)}{(+)}$ negative

f has neither a max nor a min at the critical number 0.

f takes a local maximum at 4.

Question

Consider the function $f(t) = t^4 + 4t^3$. Which of the following is true about this function?

$f'(t) = 4t^3 + 12t^2$

$= 4t^2(t+3)$

(a) f has no critical numbers.

(b) f has critical numbers 0 and -4.

(c) f has critical numbers 4 and 12.

(d) f has critical numbers 0 and -3.

Question

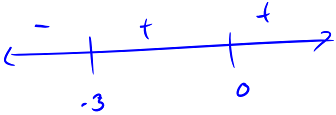
Consider the function $f(t) = t^4 + 4t^3$. Which of the following is true about this function?

(a) f has a local minimum at $t = 0$ and a local maximum at $t = -3$.

(b) f has a local minimum at $t = -3$ and a local maximum at $t = 0$.

(c) f has a local minimum at $t = -3$.

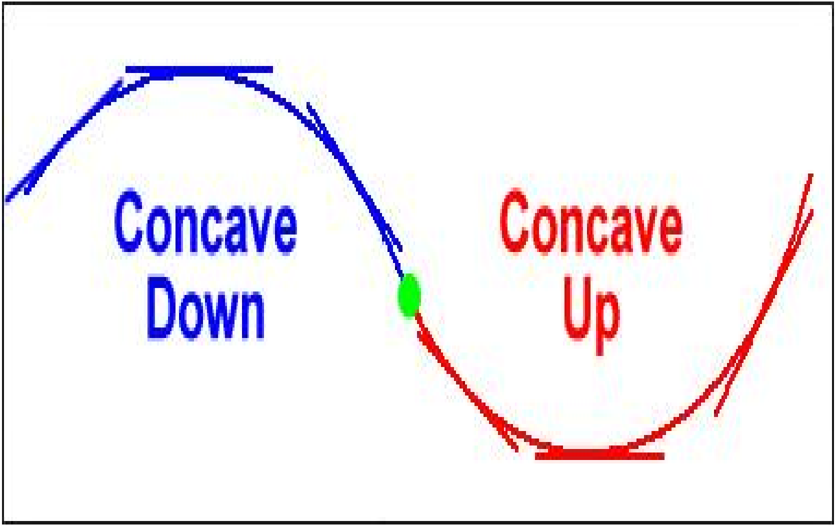
(d) f has a local minimum at $t = 0$.



Concavity and The Second Derivative

Concavity: refers to the *bending* nature of a graph. In particular, a curve is concave down if it's cupped side is down, and it is concave up if it's cupped upward.

Concavity



Figure

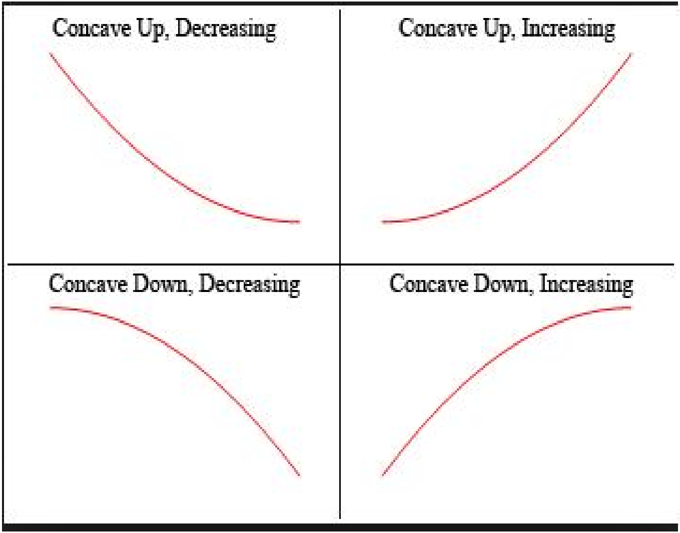


Figure: A graph can have either increasing or decreasing behavior and be either concave up or down.

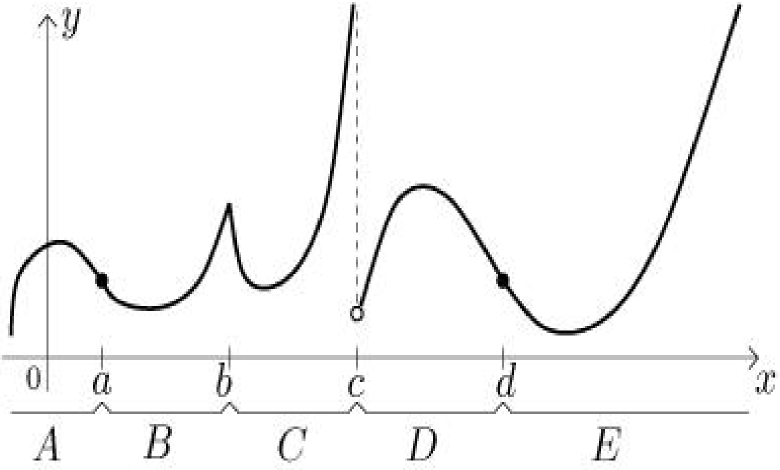


Figure: We can consider concavity at a point, but it's best thought of as a property over an interval. Many function's graphs have concavity that changes over the domain.

Definition of Concavity

Concave Up: If the graph of a function f lies above all of its tangent lines over an interval I, then f is concave up on I.

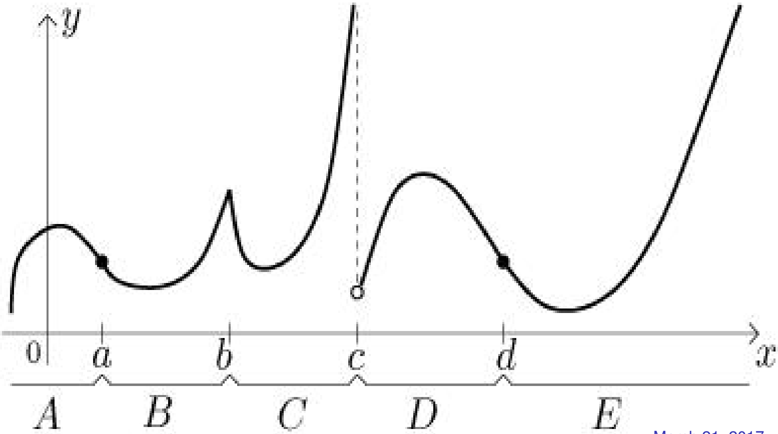
Concave Down: If the graph of f lies below each of its tangent lines on an interval I, f is concave down on I.

Theorem: (Second Derivative Test for Concavity)
Suppose f is twice differentiable on an interval I.

- If $f''(x) > 0$ on I, then the graph of f is concave up on I.

- If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Definition: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous at P and the concavity of f changes at P (from down to up or from up to down). A point where $f''(x) = 0$ would be a candidate for being an inflection point.



Concavity and Extrema:

Theorem: (Second Derivative Test for Local Extrema)
Suppose $f'(c) = 0$ and that f'' is continuous near c. Then

- if $f''(c) > 0$, f takes a local minimum at c,
- if $f''(c) < 0$, then f takes a local maximum at c.

If $f''(c) = 0$, then the test fails. f may or may not have a local extrema. You can go back to the first derivative test to find out.