March 22 Math 2306 sec. 57 Spring 2018

Section 13: The Laplace Transform

Definition: Let $f(t)$ be defined on $[0, \infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Some Examples

We computed the following Laplace transforms from the definition

$$\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$$

$$\mathscr{L}\{t\} = \frac{1}{s^2}, \quad s > 0$$

$$\mathscr{L}\{f(t)\} = \begin{cases} \frac{2}{s^2} - \frac{2}{s^2}e^{-10s} - \frac{20}{s}e^{-10s}, & s \neq 0 \\ 100, & s = 0 \end{cases}$$

where $\quad f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$

A Table of Laplace Transforms

Some basic results include:

$\mathscr{L}\{f(t)\} = F(s)$

$\mathscr{L}\{g(t)\} = G(s)$

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

Examples: Evaluate the Laplace transform of

(c) $f(t) = (2-t)^2 = 4 - 4t + t^2$

Distribute first

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{4 - 4t + t^2\} = 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$$

$$= 4\left(\frac{1}{s}\right) - 4\left(\frac{1!}{s^{1+1}}\right) + \frac{2!}{s^{2+1}}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

Examples: Evaluate

(d) $f(t) = \sin^2 5t$

Recall

$\sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$= \frac{1}{2} - \frac{1}{2}\cos(10t)$

$\mathcal{L}\{\sin^2(5t)\} = \mathcal{L}\{\frac{1}{2} - \frac{1}{2}\cos(10t)\}$

$= \frac{1}{2}\mathcal{L}\{1\} - \frac{1}{2}\mathcal{L}\{\cos(10t)\}$

$= \frac{1}{2}\left(\frac{1}{s}\right) - \frac{1}{2}\left(\frac{s}{s^2+100}\right)$

$= \frac{\frac{1}{2}}{s} - \frac{\frac{1}{2}s}{s^2+100}$

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Definition: Let $c > 0$. A function f defined on $[0, \infty)$ is said to be of *exponential order* c provided there exists positive constants M and T such that $|f(t)| < Me^{ct}$ for all $t > T$.

f grows no faster than an exponential e^{ct}.

Definition: A function f is said to be *piecewise continuous* on an interval $[a, b]$ if f has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

(no vertical asymptotes)

Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

Theorem: If f is piecewise continuous on $[0,\infty)$ and of exponential order c for some $c>0$, then f has a Laplace transform for $s>c$.

An example of a function that is NOT of exponential order for any c is $f(t)=e^{t^2}$. Note that

$$f(t)=e^{t^2}=\left(e^t\right)^t \quad \Longrightarrow \quad |f(t)|>e^{ct} \quad \text{whenever} \quad t>c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \dots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

From the table $\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = t^6$

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

Note $\frac{1}{s^7} = \frac{6!}{6!}\cdot\frac{1}{s^7} = \frac{1}{6!}\,\frac{6!}{s^7}$

so $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\,\frac{6!}{s^7}\right\} = \frac{1}{6!}\,\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}\,t^6$

Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

$$\frac{s+1}{s^2+9} = \frac{s}{s^2+3^2} + \frac{1}{s^2+3^2}$$

$$= \frac{s}{s^2+3^2} + \frac{1}{3}\,\frac{3}{s^2+3^2}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2} + \frac{1}{3}\,\frac{3}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\,\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\}$$

$$= \cos(3t) + \frac{1}{3}\sin(3t)$$

Example: Evaluate

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

we'll do a partial fraction decomp.

$$\frac{s-8}{s^2-2s} = \frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

Clear fractions

$$s-8 = A(s-2) + Bs$$

when $s=0$ $\quad -8 = -2A \quad \Rightarrow \quad A = 4$

$s=2 \quad -6 = 2B \quad \Rightarrow \quad B = -3$

$$\mathcal{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4 - 3e^{2t}$$

Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\} = \frac{2}{s^3}$?

Note

$e^{-st}e^{t} = e^{-st+t}$

By definition $\quad \mathscr{L}\left\{e^t t^2\right\} = \int_0^\infty e^{-st}e^t t^2\,dt$

$= e^{-(s-1)t}$

$= \int_0^\infty e^{-(s-1)t}\, t^2\,dt$

This is $\mathscr{L}\{t^2\}$ w/ s replaced with $s-1$.

Observe that this is simply the Laplace transform of $f(t) = t^2$ evaluated at $s-1$. Letting $F(s) = \mathscr{L}\left\{t^2\right\}$, we have

$$F(s-1) = \frac{2}{(s-1)^3}.$$

Theorem (translation in s)

Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

s^2+2s+2 is irreducible.

we'll complete the square

$$s^2+2s+2 = s^2+2s+1+1$$

$$= (s+1)^2+1$$

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1} = \frac{s+1-1}{(s+1)^2+1}$$

$$= \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

So

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}\right\}$$

$$= \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

$*$ $s+1 = s-(-1)$

and $\mathcal{L}^{-1}\left\{\frac{s}{s^2+1}\right\} = \cos t$, $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$

Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$

Use partial fractions

$$\frac{1+3s-s^2}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Clear fractions $s(s-1)^2$

$$1+3s-s^2 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$-s^2+3s+1 = (A+B)s^2 + (-2A-B+C)s + A$$

$$A = 1$$

$$-2A - B + C = 3$$

$$A + B = -1 \Rightarrow B = -1 - A = -1 - 1 = -2$$

$$C = 3 + 2A + B = 3 + 2 \cdot 1 - 2 = 3$$

$$\mathcal{L}^{-1}\left\{ \frac{1 + 3s - s^2}{s(s-1)^2} \right\} = \mathcal{L}^{-1}\left\{ \frac{1}{s} - \frac{2}{s-1} + \frac{3}{(s-1)^2} \right\}$$

$$= \mathcal{L}^{-1}\left\{ \frac{1}{s} \right\} - 2\mathcal{L}^{-1}\left\{ \frac{1}{s-1} \right\} + 3\mathcal{L}^{-1}\left\{ \frac{1}{(s-1)^2} \right\}$$

$$= 1 - 2e^t + 3e^t t$$

Note : $\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t$

so $\mathcal{L}^{-1}\left\{\frac{1}{(s-1)^2}\right\} = e^{t} \cdot t$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

Another notation is

$\mathcal{U}_a(t)$

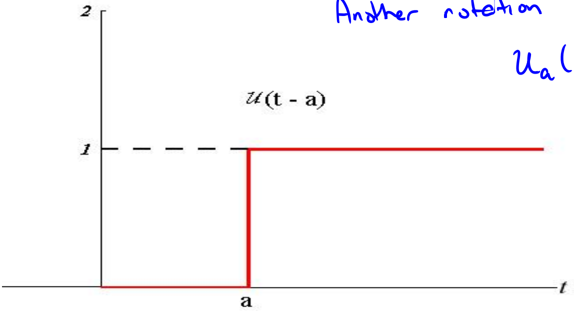

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \quad = \quad g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

For $0 \le t < a$, $\mathscr{U}(t-a) = 0$

so $f(t) = g(t) - g(t)\cdot 0 + h(t)\cdot 0 = g(t)$ as required.

For $t \ge a$, $\mathscr{U}(t-a) = 1$, then

$f(t) = g(t) - g(t)\cdot 1 + h(t)\cdot 1 = h(t)$ also as required.