# March 22 Math 2306 sec. 60 Spring 2018

## Section 13: The Laplace Transform

**Definition:** Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of $f$ is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\,dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all $s$ such that the integral is convergent.

# Some Examples

We computed the following Laplace transforms from the definition

$$\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$$

$$\mathscr{L}\{t\} = \frac{1}{s^2}, \quad s > 0$$

$$\mathscr{L}\{f(t)\} = \begin{cases} \frac{2}{s^2} - \frac{2}{s^2}e^{-10s} - \frac{20}{s}e^{-10s}, & s \neq 0 \\ 100, & s = 0 \end{cases}$$

where $\quad f(t) = \begin{cases} 2t, & 0 \le t < 10 \\ 0, & t \ge 10 \end{cases}$

# A Table of Laplace Transforms

Some basic results include:

$\mathscr{L}\{f(t)\} = F(s)$

$\mathscr{L}\{g(t)\} = G(s)$

- $\mathscr{L}\{\alpha f(t) + \beta g(t)\} = \alpha F(s) + \beta G(s)$
- $\mathscr{L}\{1\} = \frac{1}{s}, \quad s > 0$
- $\mathscr{L}\{t^n\} = \frac{n!}{s^{n+1}}, \quad s > 0$ for $n = 1, 2, \ldots$
- $\mathscr{L}\{e^{at}\} = \frac{1}{s-a}, \quad s > a$
- $\mathscr{L}\{\cos kt\} = \frac{s}{s^2+k^2}, \quad s > 0$
- $\mathscr{L}\{\sin kt\} = \frac{k}{s^2+k^2}, \quad s > 0$

# Examples: Evaluate the Laplace transform of

(a) $f(t) = \cos(\pi t)$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2+k^2}$$

$$\mathcal{L}\{\cos(\pi t)\} = \frac{s}{s^2+\pi^2}$$

# Examples: Evaluate

(b) $f(t) = 2t^4 - e^{-5t} + 3$

$$\mathcal{L}\{2t^4 - e^{-5t} + 3\} = 2\mathcal{L}\{t^4\} - \mathcal{L}\{e^{-5t}\} + 3\mathcal{L}\{1\}$$

$$= 2\left(\frac{4!}{s^{4+1}}\right) - \frac{1}{s-(-5)} + 3\left(\frac{1}{s}\right)$$

$$= \frac{48}{s^5} - \frac{1}{s+5} + \frac{3}{s}$$

# Examples: Evaluate

(c) $f(t) = (2-t)^2$ $= 4 - 4t + t^2$

expand first

$$\mathcal{L}\{(2-t)^2\} = \mathcal{L}\{4 - 4t + t^2\}$$

$$= 4\mathcal{L}\{1\} - 4\mathcal{L}\{t\} + \mathcal{L}\{t^2\}$$

$$= 4\left(\frac{1}{s}\right) - 4\left(\frac{1!}{s^{1+1}}\right) + \frac{2!}{s^{2+1}}$$

$$= \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3}$$

# Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

**Definition:** Let $c > 0$. A function $f$ defined on $[0, \infty)$ is said to be of *exponential order c* provided there exists positive constants $M$ and $T$ such that $|f(t)| < Me^{ct}$ for all $t > T$.

f can go to $\infty$ as $t \to \infty$, but no faster than an exponential.

**Definition:** A function $f$ is said to be *piecewise continuous* on an interval $[a, b]$ if $f$ has at most finitely many jump discontinuities on $[a, b]$ and is continuous between each such jump.

no vertical asymptotes

# Sufficient Conditions for Existence of $\mathscr{L}\{f(t)\}$

**Theorem:** If $f$ is piecewise continuous on $[0, \infty)$ and of exponential order $c$ for some $c > 0$, then $f$ has a Laplace transform for $s > c$.

An example of a function that is NOT of exponential order for any $c$ is $f(t) = e^{t^2}$. Note that

$$e^{ct} = \left(e^t\right)^c$$

$$f(t) = e^{t^2} = \left(e^t\right)^t \quad \Longrightarrow \quad |f(t)| > e^{ct} \quad \text{whenever} \quad t > c.$$

This is a function that doesn't have a Laplace transform. We won't be dealing with this type of function here.

# Section 14: Inverse Laplace Transforms

Now we wish to go *backwards*: Given $F(s)$ can we find a function $f(t)$ such that $\mathscr{L}\{f(t)\} = F(s)$?

If so, we'll use the following notation

$$\mathscr{L}^{-1}\{F(s)\} = f(t) \quad \text{provided} \quad \mathscr{L}\{f(t)\} = F(s).$$

We'll call $f(t)$ an inverse Laplace transform of $F(s)$.

# A Table of Inverse Laplace Transforms

- $\mathscr{L}^{-1}\left\{\frac{1}{s}\right\} = 1$
- $\mathscr{L}^{-1}\left\{\frac{n!}{s^{n+1}}\right\} = t^n$, for $n = 1, 2, \ldots$
- $\mathscr{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$
- $\mathscr{L}^{-1}\left\{\frac{s}{s^2+k^2}\right\} = \cos kt$
- $\mathscr{L}^{-1}\left\{\frac{k}{s^2+k^2}\right\} = \sin kt$

The inverse Laplace transform is also linear so that

$$\mathscr{L}^{-1}\{\alpha F(s) + \beta G(s)\} = \alpha f(t) + \beta g(t)$$

# Find the Inverse Laplace Transform

When using the table, we have to match the expression inside the brackets $\{\}$ **EXACTLY**! Algebra, including partial fraction decomposition, is often needed.

(a) $\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\}$

From the table

$$\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = t^6$$

$$\frac{1}{s^7} = \frac{6!}{6!}\,\frac{1}{s^7} = \frac{1}{6!}\,\frac{6!}{s^7}$$

$$\mathscr{L}^{-1}\left\{\frac{1}{s^7}\right\} = \mathscr{L}^{-1}\left\{\frac{1}{6!}\,\frac{6!}{s^7}\right\} = \frac{1}{6!}\,\mathscr{L}^{-1}\left\{\frac{6!}{s^7}\right\} = \frac{1}{6!}\,t^6$$

# Example: Evaluate

(b) $\mathscr{L}^{-1}\left\{\dfrac{s+1}{s^2+9}\right\}$

$$\frac{s+1}{s^2+9} = \frac{s}{s^2+3^2} + \frac{1}{s^2+3^2}$$

$$= \frac{s}{s^2+3^2} + \frac{1}{3}\,\frac{3}{s^2+3^2}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2} + \frac{1}{3}\,\frac{3}{s^2+3^2}\right\}$$

$$= \mathscr{L}^{-1}\left\{\frac{s}{s^2+3^2}\right\} + \frac{1}{3}\mathscr{L}^{-1}\left\{\frac{3}{s^2+3^2}\right\} = \cos(3t) + \frac{1}{3}\sin(3t)$$

# Example: Evaluate

we'll do a partial fraction decomp

(c) $\mathscr{L}^{-1}\left\{\dfrac{s-8}{s^2-2s}\right\}$

Clear fractions $s(s-2)$

$$\frac{s-8}{s^2-2s} = \frac{s-8}{s(s-2)} = \frac{A}{s} + \frac{B}{s-2}$$

$$s-8 = A(s-2) + Bs$$

$s=0 \qquad -8 = -2A \implies A = 4$

$s=2 \qquad -6 = 2B \implies B = -3$

$$\mathcal{L}^{-1}\left\{\frac{s-8}{s^2-2s}\right\} = \mathcal{L}^{-1}\left\{\frac{4}{s} - \frac{3}{s-2}\right\}$$

$$= 4\,\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - 3\,\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\}$$

$$= 4(1) - 3\left(e^{2t}\right)$$

$$= 4 - 3e^{2t}$$

# Section 15: Shift Theorems

Suppose we wish to evaluate $\mathscr{L}^{-1}\left\{\frac{2}{(s-1)^3}\right\}$. Does it help to know that $\mathscr{L}\left\{t^2\right\}=\frac{2}{s^3}$?

By definition $\quad \mathscr{L}\left\{e^t t^2\right\}=\int_0^\infty e^{-st}e^t t^2\,dt$

Note

$e^{-st}\cdot e^{t} = e^{-st+t}$

$= e^{-(s-1)t}$

$= \int_0^\infty e^{-(s-1)t}\, t^2\, dt$

Observe that this is simply the Laplace transform of $f(t)=t^2$ evaluated at $s-1$. Letting $F(s)=\mathscr{L}\left\{t^2\right\}$, we have

$$F(s-1)=\frac{2}{(s-1)^3}.$$

## Theorem (translation in $s$)

Suppose $\mathscr{L}\{f(t)\} = F(s)$. Then for any real number $a$

$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

# Inverse Laplace Transforms (completing the square)

(a) $\mathscr{L}^{-1}\left\{\dfrac{s}{s^2+2s+2}\right\}$

$s^2+2s+2$ is irreducible

we'll complete the square

$$s^2+2s+2 = s^2+2s+1+1$$

$$= (s+1)^2+1$$

$$\frac{s}{s^2+2s+2} = \frac{s}{(s+1)^2+1} = \frac{s+1-1}{(s+1)^2+1}$$

$$= \frac{s+1}{(s+1)^2+1} - \frac{1}{(s+1)^2+1}$$

$$\downarrow \qquad\qquad \downarrow$$

$$\frac{s}{s^2+1^2} \qquad\qquad \frac{1}{s^2+1^2}$$

Note $s+1 = s-(-1)$

$$\mathcal{L}^{-1}\left\{\frac{s}{s^2+2s+2}\right\} = \mathcal{L}^{-1}\left\{\frac{s+1}{(s+1)^2+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2+1}\right\}$$

$$= e^{-t}\cos t - e^{-t}\sin t$$

# Inverse Laplace Transforms (repeat linear factors)

(b) $\mathscr{L}^{-1}\left\{\dfrac{1+3s-s^2}{s(s-1)^2}\right\}$ Do a partial fraction decomp

Clear frac

$s(s-1)^2$

$$\frac{1+3s-s^2}{s(s-1)^2} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

$$1+3s-s^2 = A(s-1)^2 + Bs(s-1) + Cs$$

$$= A(s^2-2s+1) + B(s^2-s) + Cs$$

$$\underline{-s^2} + \underline{3s} + \underline{1} = \underline{(A+B)}s^2 + \underline{(-2A-B+C)}s \underline{+A}$$