March 22 Math 2335 sec 51 Spring 2016

Section 5.1: Numerical Integration, the Trapezoid and Simpson Rules

Our goal is to evaluate a definite integral

$$I(f) = \int_a^b f(x)\,dx$$

We may recall the Fundamental Theorem of Calculus tells us

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

provided $F(x)$ is any anti-derivative of $f(x)$.

Trapezoid Rule (one trapezoid on $[a, b]$)

$$\int_a^b P_1(x)\,dx = \frac{1}{2}(b-a)\left[f(b)+f(a)\right]$$

We'll call the right side $T_1(f)$, and we can write

$$\int_a^b f(x)\,dx \approx T_1(f).$$

The trapezoid rule with one interval is given by

$$\int_a^b f(x)\,dx \approx \frac{1}{2}(b-a)\left[f(b)+f(a)\right] = T_1(f).$$

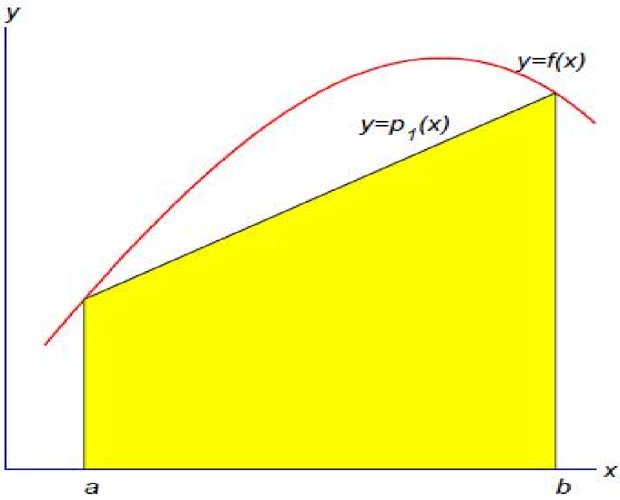

Figure: Illustration of the Trapezoid with one interval to approximate an integral.

Example

Find the approximation $T_1(f)$[1] for the integral. Compute the error and relative error.

$$T_1(f) = \frac{b-a}{2}\left[f(a) + f(b)\right]$$

$$\int_0^{0.1} e^{-x^2}\,dx$$

Here $b = 0.1$, $a = 0$, $f(x) = e^{-x^2}$

$$\int_0^{0.1} e^{-x^2}\,dx \approx T_1(f) = \frac{0.1-0}{2}\left[e^{-0^2} + e^{-(0.1)^2}\right]$$

$$= 0.05\left[1 + e^{-0.01}\right] \doteq 0.09950$$

[1]The value is $I(f) = \frac{\sqrt{\pi}}{2}\mathrm{erf}(0.1) \approx 0.09967$.

$$\mathrm{Err}(T_1(f)) = \int_0^{0.1} e^{-x^2}\,dx - T_1(f)$$

$$= 0.09967 - 0.09950 \doteq 0.00017$$

$$\mathrm{Rel}(T_1(f)) = \frac{\mathrm{Err}(T_1(f))}{\int_0^{0.1} e^{-x^2}\,dx} \doteq \frac{0.00017}{0.09967} \doteq 0.0017$$

Multiple Subintervals

We can expect to get a better approximation by dividing $[a, b]$ into several subintervals, and using a trapezoid on each.

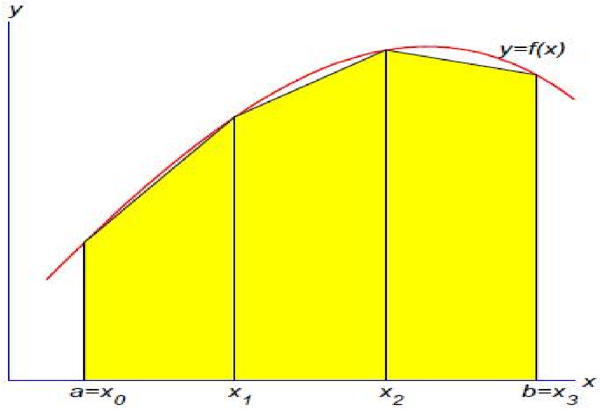

Figure: Illustration of the Trapezoid rule with three sub-intervals to approximate an integral.

Trapezoid Rule w/ n Sub-intervals

We consider an equally spaced partition of $[a, b]$

$$a = x_0 < x_1 < \cdots < x_n = b$$

where

$$x_j = x_0 + jh, \quad \text{and} \quad h = \frac{b-a}{n}.$$

By properties of integrals

$$\int_a^b f(x)\,dx = \int_{\underset{= a}{x_0}}^{x_1} f(x)\,dx + \int_{x_1}^{x_2} f(x)\,dx + \cdots + \int_{x_{n-1}}^{\overset{= b}{x_n}} f(x)\,dx$$

Trapezoid Rule w/ n Sub-intervals

Using T_1 on the interval $[x_{j-1}, x_j]$ gives

$$\int_{x_{j-1}}^{x_j} f(x)\,dx \approx T_1(f) = \frac{h}{2}[f(x_{j-1}) + f(x_j)].$$

Use this to show that

$$I(f) \approx \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

$$\int_a^b f(x)\,dx = \int_{x_0}^{x_1} \cdots + \int_{x_1}^{x_2} \cdots + \int_{x_2}^{x_3} \cdots + \cdots + \int_{x_{n-1}}^{x_n} \cdots$$

$$= \frac{h}{2}\left[f(x_0)+f(x_1)\right] + \frac{h}{2}\left[f(x_1)+f(x_2)\right] + \frac{h}{2}\left[f(x_2)+f(x_3)\right]$$

$$+ \cdots + \frac{h}{2}\left[f(x_{n-1})+f(x_n)\right].$$

$$= \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Trapezoid Rule w/ n Sub-intervals

If $[a, b]$ is divided into n equally spaced subintervals of length $h = (b-a)/n$, then $I(f) \approx T_n(f)$ where

$$T_n(f) = \frac{h}{2}[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

The T stands for "Trapezoid Rule" and the subscript n indicates the number of subintervals.

Recall $x_0 = a$, $x_j = x_0 + jh$ $j = 1, \ldots, n$

Example

Approximate $I(f)$ with $T_2(f)$ and $T_4(f)$ where

$$I(f) = \int_0^1 \frac{dx}{x^2+1}$$

Compare the answers to the exact solution $I(f) = \frac{\pi}{4}$ (recall $T_1(f) = \frac{3}{4}$).

$$T_2(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + f(x_2)\right]$$

$$h = \frac{1-0}{2} = \frac{1}{2} \qquad x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1$$

$$f(x_0) = \frac{1}{1+0^2} = 1, \quad f(x_1) = \frac{1}{1+\left(\frac{1}{2}\right)^2} = \frac{4}{5}, \quad f(x_2) = \frac{1}{1+1^2} = \frac{1}{2}$$

$$T_2(f) = \frac{1}{4}\left[1 + 2\cdot\frac{4}{5} + \frac{1}{2}\right] = \frac{1}{4}\left[\frac{10+16+5}{10}\right] = \frac{31}{40} \doteq 0.775$$

T_4 : $h = \frac{1-0}{4} = \frac{1}{4}$

$x_0 = 0$, $x_1 = \frac{1}{4}$, $x_2 = \frac{1}{2}$, $x_3 = \frac{3}{4}$, $x_4 = 1$

$$T_4(f) = \frac{h}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)\right]$$

$f(x_0) = 1$, $f(x_1) = \frac{1}{1+\left(\frac{1}{4}\right)^2} = \frac{16}{17}$, $f(x_2) = \frac{4}{5}$

$f(x_3) = \frac{1}{1+\left(3/4\right)^2} = \frac{16}{25}$, $f(x_4) = \frac{1}{2}$

$$T_4 = \frac{1}{8}\left[1 + 2\cdot\frac{16}{17} + 2\cdot\frac{4}{5} + 2\cdot\frac{16}{25} + \frac{1}{2}\right] = \frac{5323}{6800}$$

$$\doteq 0.78279$$

$$\frac{\pi}{4} \doteq 0.7854 \quad , \quad \mathrm{Rel}(T_4(f)) \doteq 0.0033$$

Simpson's Rule

Another way to improve on T_1 would be to use a higher degree polynomial—say P_2 instead of P_1. (Recall that P_2 requires three nodes.)

Special Case:
Consider $f(x)$ defined on $[-h, h]$ with the three nodes $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. We have

$$P_2(x) = f(x_0)L_0(x) + f(x_1)L_1(x) + f(x_2)L_2(x)$$

$$= f(-h)\frac{x(x-h)}{2h^2} + f(0)\frac{(h^2 - x^2)}{h^2} + f(h)\frac{x(x+h)}{2h^2}$$

Show that for $L_0(x) = x(x-h)/(2h^2)$ that

$$\int_{-h}^{h} L_0(x)\,dx = \frac{h}{3}$$

$$\int_{-h}^{h} L_0(x)\,dx = \int_{-h}^{h} \frac{x(x-h)}{2h^2}\,dx = \frac{1}{2h^2}\int_{-h}^{h}(x^2 - xh)\,dx$$

$$= \frac{1}{2h^2}\left[\frac{x^3}{3} - h\,\frac{x^2}{2}\right|_{-h}^{h}$$

$$= \frac{1}{2h^2}\left[\frac{h^3}{3} - h\,\frac{h^2}{2} - \left(\frac{-h^3}{3} - h\cdot\frac{h^2}{2}\right)\right]$$

$$= \frac{1}{2h^2}\left[\frac{h^3}{3} - \frac{h^3}{2} + \frac{h^3}{3} + \frac{h^3}{2}\right]$$

2

$$= \frac{1}{2h^2}\left[\frac{2h^3}{3}\right] = \frac{h}{3}$$

i.e. $$\int_{-h}^{h} L_0(x)\,dx = \frac{h}{3}$$

[2]It can be shown that $\int_{-h}^{h} L_2(x)\,dx = \frac{h}{3}$ as well.

Show that for $L_1(x) = (h^2 - x^2)/(h^2)$ that

$$\int_{-h}^{h} L_1(x)\,dx = \frac{4h}{3}$$

$$\int_{-h}^{h} L_1(x)\,dx = \int_{-h}^{h} \left(\frac{h^2 - x^2}{h^2}\right) dx = \frac{1}{h^2}\int_{-h}^{h} (h^2 - x^2)\,dx$$

$$= \frac{2}{h^2}\int_{0}^{h} (h^2 - x^2)\,dx$$ by even symmetry

$$= \frac{2}{h^2}\left[h^2 x - \frac{x^3}{3}\right|_0^h$$

$$= \frac{2}{h^2}\left[h^2\cdot h - \frac{h^3}{3} - 0\right]$$

$$= \frac{2}{h^2}\left[h^3 - \frac{1}{3}h^3\right] = \frac{2}{h^2}\left(\frac{2h^3}{3}\right) = \frac{4h}{3}$$

Simpson's Rule

We have $\int_{-h}^{h} L_0(x)\,dx = \int_{-h}^{h} L_2(x)\,dx = h/3$ and $\int_{-h}^{h} L_1(x)\,dx = 4h/3$ so that

$$I(f) = \int_{-h}^{h} f(x)\,dx \approx \int_{-h}^{h} P_2(x)\,dx = \frac{h}{3}[f(-h) + 4f(0) + f(h)]$$

Note that the right hand side is

$$\frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)] = S_2(f).$$

The "S" stands for Simpson's rule, and the subscript 2 indicates that there are two subintervals.

Simpson's Rule S_2 on $[a, b]$

For $I(f) = \int_a^b f(x)\,dx$, let

$$x_0 = a, \quad x_1 = \frac{a+b}{2}, \quad x_2 = b, \quad \text{and} \quad h = \frac{b-a}{2}.$$

Then

$$I(f) \approx S_2(f) = \frac{h}{3}[f(x_0) + 4f(x_1) + f(x_2)]$$

$$= \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right].$$

S_2 Illustrated

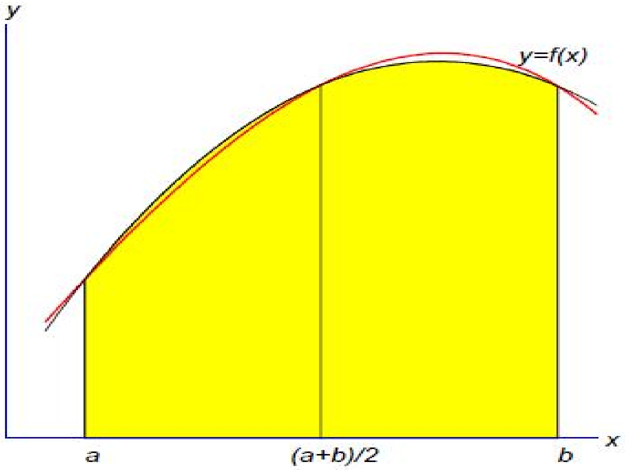

Figure: Illustration of Simpson's rule with two sub-intervals to approximate an integral.

Example

Approximate $I(f)$ with $S_2(f)$ and compare (compute error and relative error) the result to the true answer $\frac{\pi}{4}$ where

$$I(f) = \int_0^1 \frac{dx}{x^2+1} \qquad S_2(f) = \frac{h}{3}\left[f(x_0) + 4f(x_1) + f(x_2)\right],\ h = \frac{b-a}{2}$$

$$h = \frac{1-0}{2} = \frac{1}{2}, \quad x_0 = 0, \quad x_1 = \frac{1}{2}, \quad x_2 = 1$$

$$f(x_0) = 1, \quad f(x_1) = \frac{4}{5}, \quad f(x_2) = \frac{1}{2}$$

$$S_2(f) = \frac{1/2}{3}\left[1 + 4\cdot\frac{4}{5} + \frac{1}{2}\right] = \frac{1}{6}\left[1 + \frac{16}{5} + \frac{1}{2}\right] = \frac{10+32+5}{60}$$

$$= \frac{47}{60} \doteq 0.7833$$

$$Rel(S_2(f)) \doteq 0.0027$$

Recall $\quad Rel(T_4(f)) \doteq 0.0033$

Simpson's Rule with n Subintervals

The number n must be even.

Divide $[a, b]$ into n equal subintervals. Set

$$h = \frac{b-a}{n}, \quad x_0 = a, \quad x_j = a + jh, \quad j = 1, \ldots, n-1 \quad \text{and} \quad x_n = b.$$

Then $I(f) \approx S_n(f)$ where

$$S_n(f) = \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]+\frac{h}{3}[f(x_2)+4f(x_3)+f(x_4)]+$$

$$+\cdots+\frac{h}{3}[f(x_{n-2})+4f(x_{n-1})+f(x_n)].$$

Simpson's Rule with n Subintervals

$$S_n(f) = \frac{h}{3}[f(x_0)+4f(x_1)+f(x_2)]+\frac{h}{3}[f(x_2)+4f(x_3)+f(x_4)]+$$
$$+\cdots+\frac{h}{3}[f(x_{n-2})+4f(x_{n-1})+f(x_n)].$$

Note that the coefficients of $f(x_0)$ and $f(x_n)$ will be

1.

The coefficients for even numbered nodes $f(x_2)$, $f(x_4)$, etc. will be

2.

And coefficients for odd numbered nodes $f(x_1)$, $f(x_3)$, etc. will be

4.

Simpson's Rule with n Subintervals

$$S_n(f) = \frac{h}{3}[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) +$$

$$+ \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)].$$

Example

Approximate $I(f)$ with $S_4(f)$ and compare the result to the true answer $\frac{\pi}{4}$ where

$$I(f) = \int_0^1 \frac{dx}{x^2+1}$$

$$S_4 = \frac{h}{3}\left[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + f(x_4)\right]$$

$$h = \frac{1-0}{4} = \frac{1}{4} \qquad x_0 = 0,\ x_1 = \frac{1}{4},\ x_2 = \frac{1}{2},\ x_3 = \frac{3}{4},\ x_4 = 1$$

$$f(x_0) = 1,\ f(x_1) = \frac{16}{17},\ f(x_2) = \frac{4}{5},\ f(x_3) = \frac{16}{25},\ f(x_4) = \frac{1}{2}$$

$$S_2(f) = \frac{1/4}{3}\left[1 + 4\cdot\frac{16}{17} + 2\cdot\frac{4}{5} + 4\cdot\frac{16}{25} + \frac{1}{2}\right]$$

$$= \frac{1}{12}\left[1 + \frac{64}{17} + \frac{8}{5} + \frac{64}{25} + \frac{1}{2}\right] = \frac{1}{12} \cdot \frac{8011}{850}$$

$$\doteq 0.7854$$

$$Rel(S_4(f)) \doteq 0.0000076.5 = 7.65 \cdot 10^{-6}$$