March 22 Math 3260 sec. 55 Spring 2018

Section 4.4: Coordinate Systems

Definition: (Coordinate Vectors) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ where these entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

Coordinates in $\mathbb{R}^n$

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is the linear transformation defined by

$$[\mathbf{x}]_{\mathcal{B}} = P_{\mathcal{B}}^{-1}\mathbf{x}$$

where the matrix

$$P_{\mathcal{B}} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

Also, $\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}}$

Example

Let $\mathcal{B} = \left\{ \underset{\vec{b}_1}{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}, \underset{\vec{b}_2}{\begin{bmatrix} -1 \\ 1 \end{bmatrix}} \right\}$. Determine the matrix $P_{\mathcal{B}}$ and its inverse.

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix} \qquad P_{\mathcal{B}}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Use this to find

(a) the coordinate vector of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) the coordinate vector of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) a vector $\mathbf{x}$ whose coordinate vector is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

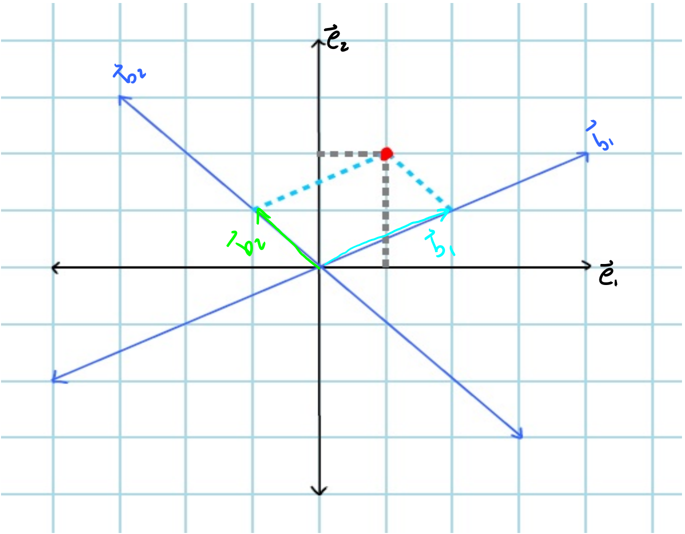

Figure: $\mathbb{R}^2$ shown using elementary basis $\{(1,0),(0,1)\}$ and with the alternative basis $\{(2,1),(-1,1)\}$.

Theorem: Coordinate Mapping

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a **one to one** mapping of V **onto** $\mathbb{R}^n$.

Remark: When such a mapping exists, we say that V is **isomorphic** to $\mathbb{R}^n$. Properties of subsets of V, such as linear dependence, can be discerned from the coordinate vectors in $\mathbb{R}^n$.

Example

Use coordinate vectors to determine if the set $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is linearly dependent or independent in $\mathbb{P}_2$.

$$\mathbf{p}(t) = 1 - 2t^2, \quad \mathbf{q}(t) = 3t + t^2, \quad \mathbf{r}(t) = 1 + t$$

We can use the standard basis $\mathcal{E} = \{1, t, t^2\}$ (in this order).

$$[\vec{p}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad [\vec{q}]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}, \quad [\vec{r}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

The polynomials are linearly dependent (independent) if the coordinate vectors are.

We can use a matrix. Let $A = \left[[\vec{p}]_{\mathcal{E}} \; [\vec{q}]_{\mathcal{E}} \; [\vec{r}]_{\mathcal{E}} \right]$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

we can use the determinant.

$$\det(A) = 1 \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} \cdots \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= -1 + 6 = 5 \neq 0$$

The columns of A are lin. independent.

Hence $\{\vec{p}, \vec{q}, \vec{r}\}$ is lin. independent in $\mathbb{P}_2$.

Section 4.5: Dimension of a Vector Space

Theorem: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set of vectors in V containing *more than n vectors* is linearly dependent.

This extends our previous result that any set of vectors (in $\mathbb{R}^n$) with more vectors than entries in each vector is automatically linearly dependent.

Dimension

Corollary: If vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every basis of V consist of exactly n vectors.

Definition: If V is spanned by a finite set, then V is called **finite dimensional**. In this case, the dimension of V

$$\dim V \;=\; \text{the number of vectors in any basis of } V.$$

The dimension of the vector space $\{\mathbf{0}\}$ containing only the zero vector is defined to be zero—i.e.

$$\dim\{\mathbf{0}\} = 0.$$

If V is not spanned by a finite set[1], then V is said to be **infinite dimensional**.

[1] $C^0(\mathbb{R})$ is an example of an infinite dimensional vector space.

Examples

(a) Find $\dim(\mathbb{R}^n)$.

The standard basis has n vectors $\vec{e}_1, \ldots, \vec{e}_n$

so $\dim(\mathbb{R}^n) = n$

(b) Determine dim Col A where $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$.

From A, a basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$ (the pivot columns)

2 vectors

$\dim \operatorname{Col} A = 2$

Some Geometry in $\mathbb{R}^3$

Give a geometric description of subspaces of $\mathbb{R}^3$ of dimension

(a) zero $\{\vec{0}\}$ The origin in $\mathbb{R}^3$.

(b) one A basis would look like $\{\vec{u}\}$ for $\vec{u}\neq\vec{0}$.

A line through the origin.

(c) two A basis is $\{\vec{u},\vec{v}\}$ with $\vec{u},\vec{v}$ lin. independent

A plane that contains the origin.

(d) three Only $\mathbb{R}^3$ itself.

Subspaces and Dimension

Theorem: Let H be a subspace of a finite dimensional vector space V. Then H is finite dimensional and

$$\dim H \leq \dim V.$$

Moreover, any linearly independent subset of H can be expanded if needed to form a basis for H.

Theorem: Let V be a vector space with dim $V = p$ where $p \geq 1$. Any linearly independent set in V containing exactly p vectors is a basis for V. Similarly, any spanning set consisting of exactly p vectors in V is necessarily a basis for V.

Column and Null Spaces

Theorem: Let A be an $m \times n$ matrix. Then

$$\dim \text{Nul}A \;=\; \text{the number of free variables in the equation } \; A\mathbf{x} = \mathbf{0},$$

and

$$\dim \text{Col}A \;=\; \text{the number of pivot positions in } \; A.$$

Example

Find the dimensions of the null and columns spaces of the matrix A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 1 & -7 & -1 \\ 3 & 0 & 6 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Pivot columns

non pivot column so x_3 would be free in $A\vec{x}=\vec{0}$

one free variable in $A\vec{x}=\vec{0}$

$\Rightarrow \dim \operatorname{Nul} A = 1$

3 pivot columns

$\Rightarrow \dim \operatorname{Col} A = 3$

Section 4.6: Rank

Definition: The **row space**, denoted Row A, of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

Example: Express the row space of A in term of a span

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix}$$

We can use the rows themselves

$$\text{Row } A = \text{Span}\left\{ \begin{bmatrix} -2 \\ -5 \\ 8 \\ 0 \\ -17 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ -5 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 3 \\ 11 \\ -19 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 7 \\ -13 \\ 5 \\ -3 \end{bmatrix} \right\}$$

* The set may or may not be a basis.

Theorem

If two matrices A and B are row equivalent, then their row spaces are the same.

In particular, if B is an echelon form of the matrix A, then the nonzero rows of B form a basis for Row B—and also for Row A since these are the same space.

Example

A matrix A along with its rref is shown.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

call this B

(a) Find a basis for Row A and state the dimension dim Row A.

A basis for Row A consists of the nonzero rows of $\underline{B}$.

A basis is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$

$\dim \text{Row} A = 3$

Example continued ...

(b) Find a basis for Col A and state its dimension.

From the rref, the pivot columns of A are the 1st, 2nd and 4th. A basis for Col A is obtained from columns 1, 2 and 4 of A.

A basis is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

$\dim \mathrm{Col}\, A = 3$