March 22 Math 3260 sec. 56 Spring 2018

Section 4.4: Coordinate Systems

Definition: (Coordinate Vectors) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an **ordered** basis of the vector space V. For each $\mathbf{x}$ in V we define the **coordinate vector of x relative to the basis** $\mathcal{B}$ to be the unique vector $(c_1, \ldots, c_n)$ in $\mathbb{R}^n$ where these entries are the weights $\mathbf{x} = c_1\mathbf{b}_1 + \cdots c_n\mathbf{b}_n$.

We'll use the notation

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = [\mathbf{x}]_{\mathcal{B}}.$$

Coordinates in $\mathbb{R}^n$

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis of $\mathbb{R}^n$. Then the change of coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_\mathcal{B}$ is the linear transformation defined by

$$[\mathbf{x}]_\mathcal{B} = P_\mathcal{B}^{-1}\mathbf{x}$$

where the matrix

$$P_\mathcal{B} = [\mathbf{b}_1 \quad \mathbf{b}_2 \quad \cdots \quad \mathbf{b}_n].$$

$$\vec{x} = P_\mathcal{B}[\vec{x}]_\mathcal{B}$$

Example

Let $\mathcal{B} = \left\{ \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{\vec{b}_1}, \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\vec{b}_2} \right\}$. Determine the matrix $P_{\mathcal{B}}$ and its inverse.

$$P_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}, \quad P_{\mathcal{B}}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}$$

Use this to find

(a) the coordinate vector of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

(b) the coordinate vector of $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$$\left[\begin{bmatrix} -1 \\ 1 \end{bmatrix}\right]_{\mathcal{B}} = \frac{1}{3}\begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

(c) a vector **x** whose coordinate vector is $[\mathbf{x}]_{\mathcal{B}} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

$$\vec{x} = P_{\mathcal{B}}[\vec{x}]_{\mathcal{B}} = \begin{bmatrix} 2 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

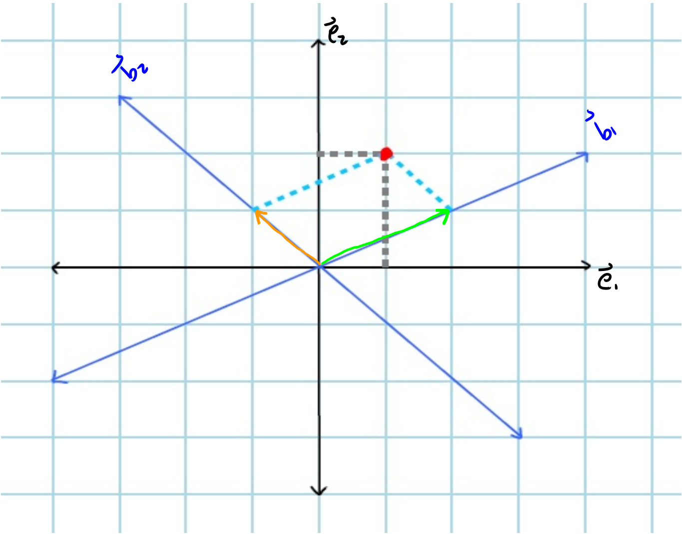

Figure: $\mathbb{R}^2$ shown using elementary basis $\{(1,0),(0,1)\}$ and with the alternative basis $\{(2,1),(-1,1)\}$.

Theorem: Coordinate Mapping

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be an ordered basis for a vector space V. Then the coordinate mapping $\mathbf{x} \mapsto [\mathbf{x}]_{\mathcal{B}}$ is a **one to one** mapping of V **onto** $\mathbb{R}^n$.

Remark: When such a mapping exists, we say that V is **isomorphic** to $\mathbb{R}^n$. Properties of subsets of V, such as linear dependence, can be discerned from the coordinate vectors in $\mathbb{R}^n$.

Example

Use coordinate vectors to determine if the set $\{\mathbf{p}, \mathbf{q}, \mathbf{r}\}$ is linearly dependent or independent in $\mathbb{P}_2$.

$$\mathbf{p}(t) = 1 - 2t^2, \quad \mathbf{q}(t) = 3t + t^2, \quad \mathbf{r}(t) = 1 + t$$

We can use the standard basis $\mathcal{E} = \{1, t, t^2\}$ in this order.

$$[\vec{p}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \quad [\vec{q}]_{\mathcal{E}} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix} \quad [\vec{r}]_{\mathcal{E}} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

We can use a matrix to determine if the coordinate vectors are lin. independent.

Let $A = \begin{bmatrix} [\vec{p}]_{\mathcal{E}} & [\vec{q}]_{\mathcal{E}} & [\vec{r}]_{\mathcal{E}} \end{bmatrix}$

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 3 & 1 \\ -2 & 1 & 0 \end{bmatrix}$$

We can use the determinant.

$$\det(A) = 1 \begin{vmatrix} 3 & 1 \\ 1 & 0 \end{vmatrix} - 0 \begin{vmatrix} \cdots \end{vmatrix} + 1 \begin{vmatrix} 0 & 3 \\ -2 & 1 \end{vmatrix}$$

$$= -1 + 6 = 5 \neq 0$$

The columns of A are lin. independent.

Hence $\{\vec{p}, \vec{q}, \vec{r}\}$ is lin. independent in $\mathbb{P}_2$.

Section 4.5: Dimension of a Vector Space

Theorem: If a vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then any set of vectors in V containing *more than n vectors* is linearly dependent.

We already know that a set of vectors (in $\mathbb{R}^n$) with more vectors than entries in each vector is lin. dependent. This is an extension of that result.

Dimension

Corollary: If vector space V has a basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$, then every basis of V consist of exactly n vectors.

Definition: If V is spanned by a finite set, then V is called **finite dimensional**. In this case, the dimension of V

$$\dim V \;=\; \text{the number of vectors in any basis of } V.$$

The dimension of the vector space $\{\mathbf{0}\}$ containing only the zero vector is defined to be zero—i.e.

$$\dim\{\mathbf{0}\} = 0.$$

If V is not spanned by a finite set[1], then V is said to be **infinite dimensional**.

[1] $C^0(\mathbb{R})$ is an example of an infinite dimensional vector space.

Examples

(a) Find $\dim(\mathbb{R}^n)$.

The standard basis for $\mathbb{R}^n$ has n vectors $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$.

$\dim(\mathbb{R}^n) = n$

(b) Determine dim Col A where $A = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & -1 \end{bmatrix}$. two pivot columns

A basis for Col A is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -1 \end{bmatrix} \right\}$

2 vectors

$\dim \text{Col} A = 2$

Some Geometry in $\mathbb{R}^3$

Give a geometric description of subspaces of $\mathbb{R}^3$ of dimension

(a) zero This is $\{\vec{0}\}$. The origin.

(b) one must be of the form $\text{span}\{\vec{u}\}$, for $\vec{u} \neq \vec{0}$.

A line through the origin

(c) two must be $\text{span}\{\vec{u}, \vec{v}\}$ with $\vec{u}, \vec{v}$ lin. independent.

A plane containing the origin

(d) three All of $\mathbb{R}^3$

Subspaces and Dimension

Theorem: Let H be a subspace of a finite dimensional vector space V. Then H is finite dimensional and

$$\dim H \le \dim V.$$

Moreover, any linearly independent subset of H can be expanded if needed to form a basis for H.

Theorem: Let V be a vector space with dim $V = p$ where $p \ge 1$. Any linearly independent set in V containing exactly p vectors is a basis for V. Similarly, any spanning set consisting of exactly p vectors in V is necessarily a basis for V.

Column and Null Spaces

Theorem: Let A be an $m \times n$ matrix. Then

$$\dim \text{Nul}A \;=\; \text{the number of free variables in the equation } \; A\mathbf{x} = \mathbf{0},$$

and

$$\dim \text{Col}A \;=\; \text{the number of pivot positions in } \; A.$$

Example

Find the dimensions of the null and columns spaces of the matrix A.

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ -3 & 1 & -7 & -1 \\ 3 & 0 & 6 & 1 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

pivot columns

not pivot column

x_3 would be the only free variable for $A\vec{x} = \vec{0}$

$\dim \operatorname{Nul} A = 1$

one free variable for $A\vec{x} = \vec{0}$

$\dim \operatorname{Col} A = 3$ 3 pivot columns.