March 23 Math 1190 sec. 62 Spring 2017

Section 4.4: Local Extrema and Concavity

Let's start with some review question inspired by this Thursday's quiz!

Question

The limit $\lim_{x \to 1} \frac{\sin(\pi x)}{\ln x}$ gives rise to

(a) the indeterminate form $\frac{0}{0}$.

(b) the indeterminate form $\frac{\infty}{\infty}$.

(c) the indeterminate form $\infty - \infty$.

(d) no indeterminate form, the limit is 1.

$\sin(\pi) = 0$

and

$\ln 1 = 0$

Question

Evaluate $\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x}$ (use any applicable technique)

Use l'H rule

(a) $\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x} = 1$

$$\lim_{x\to 1} \frac{\cos(\pi x)\cdot\pi}{\frac{1}{x}} = \frac{\cos(\pi)\cdot\pi}{\frac{1}{1}}$$

(b) $\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x} = -\pi$

(c) $\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x} = \infty$

$$= \frac{(-1)\pi}{1} = -\pi$$

(d) $\lim_{x\to 1} \frac{\sin(\pi x)}{\ln x} = 0$

Question

The critical numbers of the function $f(x) = x^3 - 12x$ are

(a) -2 and 2.

$f'(x) = 3x^2 - 12 = 3(x^2-4)$

(b) -2, 0, and -2.

$f'(x)$ is undefined ... never

(c) nothing since f has no critical numbers.

$f'(x) = 0$

(d) 0 and 4.

$3(x^2-4) = 0$

$\Rightarrow x^2 = 4 \Rightarrow x = \pm 2.$

Question

Determine the absolute maximum and absolute minimum values of $f(x) = x^3 - 12x$ on the interval $[0, 3]$.

Check

(a) The max is 15, and the min is -12.

$f(0) = 0 \leftarrow$ max

(b) The max is 0, and the min is -9.

$f(2) = -16 \leftarrow$ min

(c) The max is 0, and the min is -16.

$f(3) = -9$

(d) The max is 0, and there is no absolute minimum.

Our Latest Theorems & Definitions

Theorem: First Derivative Test: Let f be continuous and suppose that c is a critical number of f.

- If f' changes from$-$ to $+$ at c, then f has a local minimum at c.
- If f' changes from $+$ to $-$ at c, then f has a local maximum at c.
- If f' does not change signs at c, then f does not have a local extremum at c.

Definition: Concavity

Concave Up: If the graph of a function f lies above all of its tangent lines over an interval I, then f is concave up on I.

Concave Down: If the graph of f lies below each of its tangent lines on an interval I, f is concave down on I.

Our Latest Theorems & Definitions

Theorem: (Second Derivative Test for Concavity)
Suppose f is twice differentiable on an interval I.

- If $f''(x) > 0$ on I, then the graph of f is concave up on I.
- If $f''(x) < 0$ on I, then the graph of f is concave down on I.

Definition: A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous at P and the concavity of f changes at P (from down to up or from up to down).

A point where $f''(x) = 0$ would be a candidate for being an inflection point.

Example

Determine where the graph of $f(x) = x^4 - 4x^3$ is concave up, where it is concave down, and identify any x-values at which f has a point of inflection.

We'll do a sign analysis on f''.

Domain of f is $(-\infty, \infty)$

$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$

$f''(x) = 12x^2 - 24x = 12x(x-2)$

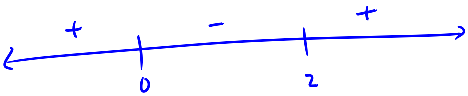
f'' is defined everywhere

$f''(x) = 0 \implies 12x(x-2) = 0$

$\implies x = 0$ or $x - 2 = 0 \implies x = 2$

we'll break the domain into $(-\infty, 0)$, $(0, 2)$, $(2, \infty)$.

$$f''(x) = 12x\,(x-2)$$



sign of f''

test pt. -1 $\quad f''(-1) = 12(-1)(-1-2)$

1 $\quad f''(1) = 12(1)(1-2)$

3 $\quad f''(3) = 12(3)(3-2)$

f is concave up on $(-\infty, 0) \cup (2, \infty)$

f is concave down on $(0, 2)$.

f has inflection points @ $x = 0$ and $x = 2$.

Concavity and Extrema:

Theorem: (Second Derivative Test for Local Extrema)
Suppose $f'(c) = 0$ and that f'' is continuous near c. Then

- if $f''(c) > 0$, f takes a local minimum at c,

- if $f''(c) < 0$, then f takes a local maximum at c.

If $f''(c) = 0$, then the test fails. f may or may not have a local extrema. You can go back to the first derivative test to find out.

Example: Consider $f(x) = x \ln x$

f has one local extreme value. Find the local extreme value and classify it as a local maximum or local minimum using the second derivative test.

The domain of f is $(0, \infty)$.

Find crit. #'s

$$f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1$$

f' is defined on $(0, \infty)$.

$$f'(x) = 0 \Rightarrow \ln x + 1 = 0 \Rightarrow \ln x = -1$$

$$e^{\ln x} = e^{-1} \Rightarrow x = e^{-1}$$ our only critical #

$f''(x) = \frac{1}{x}$ crit. # e^{-1}

2nd der. test

$$f''(e^{-1}) = \frac{1}{e^{-1}} = e^{1} = e$$

$$f''(e^{-1}) > 0$$

f is concave up @ $x = e^{-1}$, f takes a local min there.

The local minimum value is $f(e^{-1}) = e^{-1} \ln e^{-1} = -e^{-1}$

Let's Analyze a Function

Consider the function $f(x) = xe^{3x}$. Let's determine

(a) the intervals on which f is increasing and decreasing,
(b) the intervals on which f is concave up and concave down,
(c) identify critical points and classify any local extrema, and
(d) identify any points of inflection.

Question

Find the first and second derivatives of $f(x) = xe^{3x}$.

$f'(x) = 1 \cdot e^{3x} + x\,e^{3x} \cdot 3$

$= (1+3x)\,e^{3x}$

$f''(x) = 3e^{3x} + 1 \cdot e^{3x} \cdot 3 + x\,e^{3x} \cdot 3^2$

$= (6+9x)\,e^{3x}$

(a) $f'(x) = 3e^{3x}, \quad f''(x) = 9e^{3x}$

(b) $f'(x) = (1+3x)e^{3x}, \quad f''(x) = (6+9x)e^{3x}$

(c) $f'(x) = (1+3x)e^{3x}, \quad f''(x) = (1+3x)e^{3x}$

(d) $f'(x) = e^{3x}+3x^2e^{2x}, \quad f''(x) = 3e^{3x}+6xe^{2x}+6x^3e^x$

Question

We found for $f(x) = xe^{3x}$ that $f'(x) = (1 + 3x)e^{3x}$. The critical number(s) of f are

The domain of f is $(-\infty, \infty)$.

(a) $\frac{1}{3}$ only

f' is always defined

(b) $\frac{1}{3}$ and 0

$f'(x) = 0 \Rightarrow (1+3x)e^{3x} = 0$

(c) $-\frac{1}{3}$ only

$1+3x=0$ or $e^{3x}=0$

(d) $-\frac{1}{3}$ and 0

$x = \frac{-1}{3}$ no soln.

$-1/3$

Question

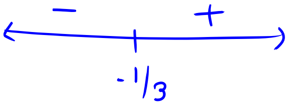
We found for $f(x) = xe^{3x}$ that $f'(x) = (1 + 3x)e^{3x}$. And f has one critical number $-\frac{1}{3}$. f has the increasing/decreasing behavior

(a) f is decreasing on $\left(-\infty, -\frac{1}{3}\right)$ and increasing on $\left(-\frac{1}{3}, \infty\right)$

(b) f is increasing on $\left(-\infty, -\frac{1}{3}\right)$ and decreasing on $\left(-\frac{1}{3}, \infty\right)$

(c) f is increasing on $(-\infty, \infty)$

(d) f is decreasing on $(-\infty, \infty)$



Question

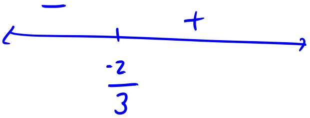
We found for $f(x) = xe^{3x}$ that $f''(x) = 3(2+3x)e^{3x}$. And $f''(x)$ has one root $-\frac{2}{3}$. f has the concavity

(a) f is concave up on $(-\infty, -\frac{2}{3})$ and concave down on $(-\frac{2}{3}, \infty)$

(b) f is concave down on $(-\infty, -\frac{2}{3})$ and concave up on $(-\frac{2}{3}, \infty)$

(c) f is concave up on $(-\infty, \infty)$

(d) f is concave down on $(-\infty, \infty)$



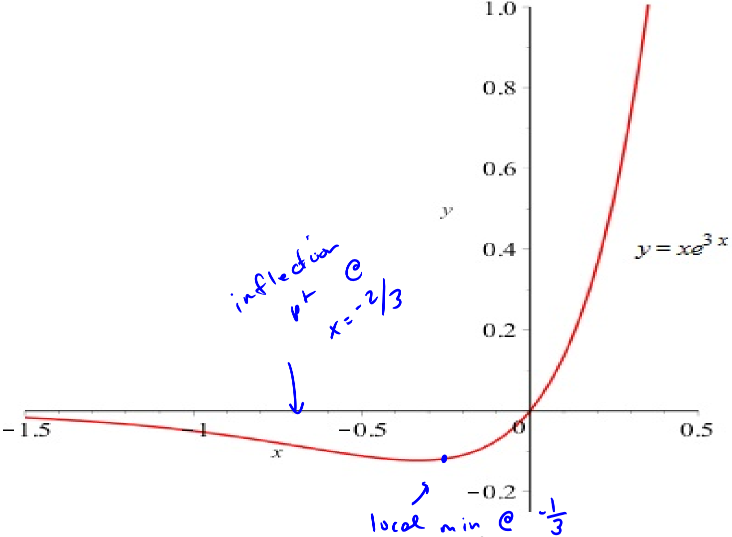
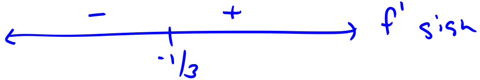


Figure: Plot of $y = xe^{3x}$.

$f(x) = xe^{3x}$

$f'(x) = (1+3x)e^{3x}$ and $f''(x) = 3(2+3x)e^{3x}$. f has one critical number $-1/3$. Observe how it can be classified as a local minimum.

1st der. test



$f'(x) < 0$ to the left and $f'(x) > 0$ to the right of $-\frac{1}{3}$.

we have a local min.

2nd der. test.

$$f''\left(-\tfrac{1}{3}\right) = 3\left(2+3\left(-\tfrac{1}{3}\right)\right)e^{3\left(-\frac{1}{3}\right)}$$

$$= 3(1)e^{-1} > 0$$

we have a local minimum.

Section 4.8: Antiderivatives; Differential Equations

We have a bunch of rules for taking derivatives, now we'll think about the other direction—if we know $f'(x)$, can we find f?

Definition: A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

For example if $f(x) = 2x$, an antiderivative is

$$F(x) = x^2 .$$

General Antiderivative

Recall: The MVT told us that if $f'(x) = g'(x)$ on an interval, then they are equal except up to an added constant—i.e. $f(x) = g(x) + C$ for some constant C. We'll use this again.

Theorem: If F is any antiderivative of f on an interval I, then the *most general* antiderivative of f on I is

$$F(x) + C \quad \text{where } C \text{ is an arbitrary constant.}$$

e.g. for $f(x) = 2x$, we'd have $x^2 + C$

Find the most general antiderivative of f.

(i) $f(x) = 2e^{2x}, \quad I = (-\infty, \infty)$

an antider. is e^{2x}

So the most general is $F(x) = e^{2x} + C$

for constant C.

(ii) $f(x) = \dfrac{1}{x}, \quad I = (0, \infty)$

$F(x) = \ln x + C$

Question: Find the most general antiderivative of f.

(iii) $f(x) = \sin x, \quad I = (-\infty, \infty)$

(a) $F(x) = \cos x$

(b) $F(x) = \cos x + C$

(c) $F(x) = -\cos x + C$

Question: Find the most general antiderivative of f.

(iv) $f(x) = \sec x \tan x, \quad I = \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(a) $F(x) = \sec x$

(b) $F(x) = \sec x + C$

(c) $F(x) = \tan x + C$

Find the most general antiderivative of

$$f(x) = x^n, \quad \text{where } n = 1, 2, 3, \ldots$$

Let's guess that $F(x) = A x^k$ for some constant A and power k.

we need $F'(x) = f(x)$

$$Akx^{k-1} = x^n = 1x^n$$

Matching powers and coefficients

$$k-1 = n \implies k = n+1$$

and $\quad Ak = 1 \implies A(n+1) = 1$

$$A = \frac{1}{n+1}$$

An antiderivative is $\quad \dfrac{1}{n+1} x^{n+1}$

The most general is $\quad \dfrac{1}{n+1} x^{n+1} + C$

Check: is $\dfrac{d}{dx}\left(\dfrac{1}{n+1} x^{n+1} + C\right) = x^n$?

$$\frac{d}{dx}\left(\frac{1}{n+1} x^{n+1} + C\right) = \frac{1}{n+1}(n+1) x^{n+1-1} + 0 = x^n \quad \checkmark$$

Some general results[1]:

(See the table on page 330 in Sullivan & Miranda for a more comprehensive list.)

Function	Particular Antiderivative	Function	Particular Antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sin x$	$-\cos x$
x^n, $n \neq -1$	$\frac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\ln\lvert x\rvert$	$\csc x \cot x$	$-\csc x$
$\frac{1}{x^2+1}$	$\tan^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$

[1] We'll use the term particular antiderivative to refer to any antiderivative that has no arbitrary constant in it.