Mar. 23 Math 2254H sec 015H Spring 2015

Section 11.4: Comparison Tests

Note: In this section, we restrict our attention to series of nonnegative terms.

Motivating Example: Consider the two similar—yet different—series:

$$\text{(i)} \quad \sum_{n=0}^{\infty} \frac{1}{3^n} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{n=0}^{\infty} \frac{1}{3^n + 7}$$

Question: Does the series on the right, (ii), converge or diverge?

Comparing Series

$$\text{(i)} \quad \sum_{n=0}^{\infty} \frac{1}{3^n} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{n=0}^{\infty} \frac{1}{3^n+7}$$

- The one on the left, (i), is geometric $|r| = |1/3| < 1$—obviously convergent.
- The one on the right, (ii), is not geometric. And $f(x) = \frac{1}{3^x+7}$ isn't readily integrated!
- Does it help to notice that

$$\frac{1}{3^n+7} < \frac{1}{3^n}$$

for every value of n?

Let $\{s_n\}$ be the sequence of partial sums for $\sum_{n=0}^{\infty} \frac{1}{3^n}$, and let $\{t_n\}$ be the sequence of partial sums for $\sum_{n=0}^{\infty} \frac{1}{3^n+7}$.

$s_0 = \frac{1}{3^0}$

$s_1 = \frac{1}{3^0} + \frac{1}{3}$

$s_2 = \frac{1}{3^0} + \frac{1}{3} + \frac{1}{3^2}$

$\vdots$

$s_{n+1} = s_n + \frac{1}{3^{n+1}} > s_n$

$\{s_n\}$ is increasing.

Also, $s_n \to \frac{1}{1-\frac{1}{3}} = \frac{3}{2}$

Note $\{s_n\}$ is bounded

$0 \le s_n \le \frac{3}{2}$ for all n

$$t_0 = \frac{1}{3^0+7} < S_0$$

$$t_1 = \frac{1}{3^0+7} + \frac{1}{3^1+7} < S_1$$

$$t_2 = \frac{1}{3^0+7} + \frac{1}{3^1+7} + \frac{1}{3^2+7} < S_2$$

$\vdots$

In fact $t_n \le S_n$ for all $n \geq 0$.

$t_{n+1} \geq t_n$ for all n.

$\{t_n\}$ is bounded and monotonic.

Hence t_n converges.

The series is convergent.

Comparing Series

What if we consider the two series

$$\text{(i)} \quad \sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n \qquad \text{and} \qquad \text{(ii)} \quad \sum_{n=1}^{\infty} \left[\ln(n)\left(\frac{5}{2}\right)^n\right]$$

- The one on the left, (i), is geometric $|r| = |5/2| > 1$—obviously divergent.
- The one on the right, (ii), is not geometric. And $f(x) = \ln(x)(5/2)^x$ isn't readily integrated!
- And for every $n \geq 3$,

$$\ln(n)\left(\frac{5}{2}\right)^n > \left(\frac{5}{2}\right)^n.$$

If $\{s_n\}$ is the sequence of partial sums for $\sum_{n=1}^{\infty} \left(\frac{5}{2}\right)^n$. $\lim_{n\to\infty} s_n = \infty$

If $\{t_n\}$ is the sequence of partial sums for $\sum_{n=1}^{\infty} \left(\ln(n) \left(\frac{5}{2}\right)^n \right)$

$$t_{n+2} \geq s_{n+2}$$

$$\lim_{n\to\infty} t_n = \infty$$

Hence the series is divergent.

Theorem: The Direct Comparison Test

Theorem: Suppose $\sum a_n$ and $\sum b_n$ are series of positive terms, such that[1]

$$a_n \ \leq \ b_n \quad \text{for each } n.$$

(i) If $\sum b_n$ is convergent, then $\sum a_n$ is convergent.

(ii) If $\sum a_n$ is divergent, then $\sum b_n$ is dinvergent.

[1] If the condition $a_n \leq b_n$ doesn't hold for some first finite number of terms, the result is unchanged. We could say $a_n \leq b_n$ for all $n \geq n_0$ for some number n_0.

Example

Determine the convergence or divergence of the series.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n+7}$ Convergent

(b) $\displaystyle\sum_{n=1}^{\infty} \left[\ln(n)\left(\frac{5}{2}\right)^n\right]$ Divergent.

Example

Determine the convergence or divergence of the series.

(c) $$\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^7 + 4n^2 + 3}}$$

Note $n^7 + 4n^2 + 3 \geq n^7 \implies$

$$\frac{1}{n^7} \geq \frac{1}{n^7+4n^2+3} \implies$$

$$\frac{1}{\sqrt[3]{n^7}} \geq \frac{1}{\sqrt[3]{n^7+4n^2+3}} \implies$$

$$\frac{n}{n^{7/3}} \geq \frac{n}{\sqrt[3]{n^7+4n^2+3}}$$

$$\Longrightarrow \quad \frac{1}{n^{4/3}} \geq \frac{n}{\sqrt[3]{n^7+4n^2+3}}$$

$\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ is a p-series w/ $p=\frac{4}{3}>1$

which is convergent.

Hence $\sum_{n=1}^{\infty} \frac{n}{\sqrt[3]{n^7+4n^2+3}}$ is

convergent by direct comparison.

Example

Determine the convergence or divergence of the series.

(d) $\displaystyle\sum_{n=2}^{\infty} \frac{n^3}{n^4 - 2}$

Note $n^4 - 2 \leq n^4$

$$\frac{1}{n^4} \leq \frac{1}{n^4-2} \quad \text{for } n \geq 2$$

$$\frac{n^3}{n^4} \leq \frac{n^3}{n^4-2}$$

$$\Rightarrow \quad \frac{1}{n} \leq \frac{n^3}{n^4-2}$$

Since $\sum_{n=2}^{\infty} \frac{1}{n}$ diverges,

$\sum_{n=2}^{\infty} \frac{n^3}{n^4-2}$ diverges by

direct comparison.

A Potential Fly in the Ointment

Consider the two series

$$\text{(i)} \quad \sum_{n=2}^{\infty} \frac{1}{3^n} \qquad \text{and} \qquad \text{(ii)} \quad \sum_{n=2}^{\infty} \frac{1}{3^n-7}$$

$$\text{Unfortunately} \quad \frac{1}{3^n-7} \not\leq \frac{1}{3^n}. \quad \text{in fact} \quad \frac{1}{3^n} \leq \frac{1}{3^n-7}.$$

Don't we strongly suspect that $\sum \frac{1}{3^n-7}$ *ought* to be convergent? It is SO similar to $\sum \frac{1}{3^n}$.

We need a way to compare **similar** series that doesn't require such a specific inequality!

Theorem: The Limit Comparison Test

Theorem: Suppose $\sum a_n$ and $\sum b_n$ are series of positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c, \quad \text{and} \quad 0 < c < \infty,$$

Then either both series converge, or both series diverge.

Example

Determine the convergence or divergence of the series.

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$

For $n \to \infty$ $n\sqrt{n^2-1}$ behaves like n^2.

Use limit comparison test w/ the convergent p-series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$.

Let $a_n = \dfrac{1}{n\sqrt{n^2-1}}$ and $b_n = \dfrac{1}{n^2}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n\sqrt{n^2-1}}}{\frac{1}{n^2}}$$

$$= \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{n^2}{n^2\sqrt{1-\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}}$$

$$= 1 \quad , \qquad 0 < 1 < \infty$$

Hence $\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ converges.

Example

Determine the convergence or divergence of the series.

(b) $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{5n^3 - 2n + 1}}$

For $n \to \infty$

$\sqrt[4]{5n^3 - 2n+1}$ behaves like $\sqrt[4]{5n^3}$

Limit comparison w/ the known divergent series $\sum_{n=1}^{\infty} \frac{1}{n^{3/4}}$.

Let $a_n = \frac{1}{n^{3/4}}$ and $b_n = \frac{1}{\sqrt[4]{5n^3 - 2n+1}}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{n^{3/4}}}{\frac{1}{\sqrt[4]{5n^3-2n+1}}}$$

$$= \lim_{n\to\infty} \frac{\sqrt[4]{5n^3-2n+1}}{n^{3/4}}$$

$$= \lim_{n\to\infty} \sqrt[4]{\frac{5n^3-2n+1}{n^3}} = \lim_{n\to\infty} \sqrt[4]{5 - \frac{2}{n^2} + \frac{1}{n^3}}$$

$$= \sqrt[4]{5}, \quad 0 < \sqrt[4]{5} < \infty.$$

So $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{5n^3-2n+1}}$ diverges.

Analyzing Expressions with Powers and Roots

Identify the **leading term** in the numerator and in the denominator. Then take the ratio. For example:

$$\frac{\sqrt{3n^3+4n-6}}{\sqrt[5]{8n^{12}+32n^7-6n^4+12}} \sim \frac{\sqrt{3n^3}}{\sqrt[5]{8n^{12}}}$$

$$\frac{n^{3/2}}{n^{12/5}} = \frac{1}{n^{\frac{12}{5}-\frac{3}{2}}} = \frac{1}{n^{9/10}}$$

Using a Comparison Test

- First try to determine if you think a series converges or diverges.
- Next, pick a series to compare it to such that **(1)** this series has the same convergence/divergence behavior, and **(2)** you can prove it! (usually use a p-series).
- Take the limit if using limit comparison—it doesn't matter who you call a_n and who you call b_n.
- Set up the inequality ($a_n \leq b_n$) if using direct comparison. If your series converges, it should be a_n. If your series diverges, you want it to be b_n.
- **Clearly state your final conclusion for all the world to see!**