March 24 Math 2306 sec 58 Spring 2016

Section 10: Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x)+u_2(x)y_2(x)$

We found that the functions u_1 and u_2 were determined as

$$u_1(x) = \int \frac{-y_2(x)g(x)}{W}\,dx \quad \text{and} \quad u_2(x) = \int \frac{y_1(x)g(x)}{W}\,dx$$

where W is the Wronskian of y_1 and y_2.

Example:

Solve the ODE

$$x^2y'' + xy' - y = x^2,$$

given that $y_1 = x$ is one solution to the associated homogeneous equation.

homogeneous eqn $\quad x^2y'' + xy' - y = 0$

Standard form $\quad y'' + \frac{1}{x}y' - \frac{1}{x^2}y = 0$

Find y_2 using reduction of order.

$y_2 = u(x)\,y_1(x)$ where $u(x) = \displaystyle\int \frac{e^{-\int P(x)\,dx}}{(y_1(x))^2}\,dx$

$P(x) = \frac{1}{x}$ $\qquad -\int P(x)\,dx = -\int \frac{1}{x}\,dx = -\ln|x|$

$$= \ln x^{-1}$$

assuming $x > 0$

$$u = \int \frac{e^{\ln x^{-1}}}{(x)^2}\,dx = \int \frac{x^{-1}}{x^2}\,dx = \int x^{-3}\,dx = -\frac{1}{2}x^{-2}$$

so $\quad y_2 = u y_1 = \frac{-1}{2}x^{-2}\cdot x = \frac{-1}{2}x^{-1}$

By superposition, we can take $y_2 = x^{-1}$

$$x^2 y'' + x y' - y = x^2, \qquad \text{with } y_1 = x,\ y_2 = x^{-1}$$

Standard form:

$$y'' + \frac{1}{x} y' - \frac{1}{x^2} y = 1 \quad \Longrightarrow \quad g(x) = 1$$

Compute the Wronskian

$$W = \begin{vmatrix} x & x^{-1} \\ 1 & -x^{-2} \end{vmatrix} = x(-x^{-2}) - 1 \cdot x^{-1} = -x^{-1} - x^{-1} = -2x^{-1}$$

$$u_1 = \int \frac{-y_2(x)\,g(x)}{W}\,dx = \int \frac{-x^{-1}\cdot 1}{-2x^{-1}}\,dx$$

$$= \frac{1}{2}\int \frac{x^{-1}}{x^{-1}}\,dx = \frac{1}{2}\int dx = \frac{1}{2}x$$

$$u_2 = \int \frac{y_1(x)\,g(x)}{W}\,dx = \int \frac{x\cdot 1}{-2x^{-1}}\,dx = \frac{-1}{2}\int x\cdot x\,dx$$

$$= \frac{-1}{2}\int x^2\,dx = \frac{-1}{2}\cdot\frac{x^3}{3} = \frac{-x^3}{6}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \frac{1}{2}x\,(x) - \frac{x^3}{6}\left(x^{-1}\right)$$

$$= \frac{1}{2}x^2 - \frac{1}{6}x^2 = \left(\frac{3}{6} - \frac{1}{6}\right)x^2 = \frac{2}{6}x^2 = \frac{1}{3}x^2$$

The general solution to the ODE is

$$\boxed{y = y_c + y_p\,, \quad y = c_1 x + c_2 x^{-1} + \frac{1}{3}x^2}$$

Solve the IVP

$$x^2y'' + xy' - y = x^2, \quad y(1) = -1, \quad y'(1) = 0$$

From the previous example the general soln. to the ODE is

$$y = c_1 x + c_2 x^{-1} + \frac{1}{3}x^2$$

$$y' = c_1 - c_2 x^{-2} + \frac{2}{3} x$$

$$y(1) = -1 = C_1 \cdot 1 + C_2\left(\frac{1}{1}\right) + \frac{1}{3}(1)^2 \Rightarrow C_1 + C_2 + \frac{1}{3} = -1$$

$$C_1 + C_2 = \frac{-4}{3}$$

$$y'(1) = 0 = C_1 - C_2\left(\frac{1}{1^2}\right) + \frac{2}{3}\cdot 1 \Rightarrow C_1 - C_2 + \frac{2}{3} = 0$$

$$C_1 - C_2 = \frac{-2}{3}$$

$$\begin{aligned} C_1 + C_2 &= \frac{-4}{3} \\ C_1 - C_2 &= \frac{-2}{3} \end{aligned}$$

add

$$2C_1 = \frac{-6}{3} = -2 \Rightarrow C_1 = -1$$

$$C_2 = \frac{-4}{3} - C_1 = \frac{-4}{3} - (-1) = \frac{-1}{3}$$

The solution to the IVP is

$$y = -x - \frac{1}{3}x^{-1} + \frac{1}{3}x^2$$

Simple Harmonic Motion

We consider a flexible spring from which a mass is suspended. In the absence of any damping forces (e.g. friction, a dash pot, etc.), and free of any external driving forces, any initial displacement or velocity imparted will result in **free, undamped motion**–a.k.a. **simple harmonic motion.**

Harmonic Motion gif

Building an Equation: Hooke's Law

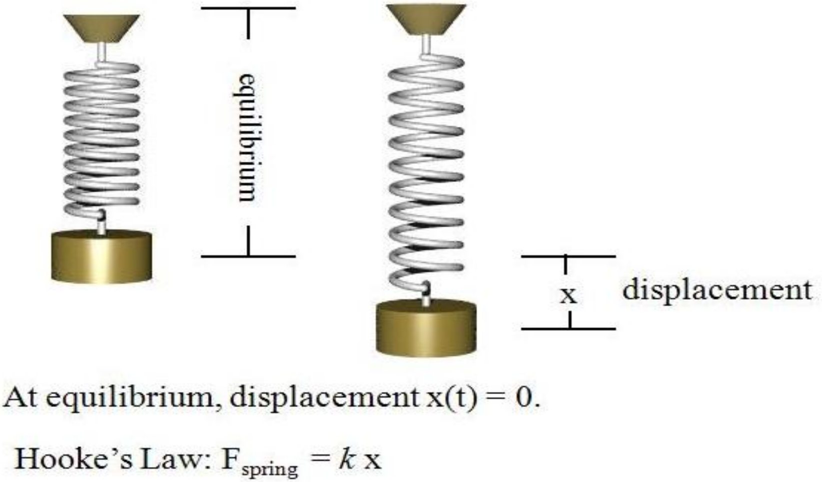

Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ Force = mass times acceleration

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ Force exerted by the spring is proportional to displacement
The force imparted by the spring opposes the direction of motion.

$mx'' + kx = 0 \Rightarrow x'' + \frac{k}{m}x = 0$

$$m\frac{d^2x}{dt^2} = -kx \quad \Longrightarrow \quad x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Convention We'll Use: Up will be positive ($x > 0$), and down will be negative ($x < 0$). This orientation is arbitrary and follows the convention in Trench.

Obtaining the Spring Constant (US Customary Units)

If an object with weight W pounds stretches a spring δx feet[1] from it's length with no mass attached, then by Hooke's law we compute the spring constant via the equation

$$W = k\delta x.$$

The units for k in this system of measure are lb/ft.

$$W \text{ lb} = k\ \delta x \text{ ft} \quad \Longrightarrow \quad k = \frac{W}{\delta x}\ \frac{\text{lb}}{\text{ft}}$$

[1]Note that $\delta x =$ w/ mass equilibrium - w/o mass equilibrium.

Obtaining the Spring Constant (US Customary Units)

Note also that Weight = mass $\times$ acceleration due to gravity. Hence if we know the weight of an object, we can obtain the mass via

$$W = mg.$$

We typically take the approximation $g = 32$ ft/sec^2. The units for mass are lb sec^2/ft which are called slugs.

$$W \text{ lb} = m \, g \, \frac{\text{ft}}{\text{sec}^2} \implies m = \frac{W}{g} \frac{\text{lb}}{\text{ft}/\text{sec}^2} = \frac{W}{g} \frac{\text{lb sec}^2}{\text{ft}}$$

Obtaining the Spring Constant (SI Units)

In SI units, the weight would be expressed in Newtons (N). The appropriate units for displacement would be meters (m). In these units, the spring constant would have units of N/m.

It is customary to describe an object by its mass in kilograms. When we encounter such a description, we deduce the weight in Newtons

$$W = mg \quad \text{taking the approximation} \quad g = 9.8\,\text{m}/\text{sec}^2.$$

Obtaining the Spring Constant: *Displacment in Equilibrium*

If an object stretches a spring δx units from it's length (with no object attached), we may say that it stretches the spring δx units *in equilibrium*. Applying Hooke's law with the weight as force, we have

$$mg = k\delta x.$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

Provided that values for δx and g are used in appropriate units, ω is in units of per second.

Simple Harmonic Motion

Characteristic eqn.

$r^2 + \omega^2 = 0 \Rightarrow r^2 = -\omega^2$

$r = \pm i\omega$

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion.**

Caution: The phrase *equation of motion* is used differently by different authors. Some, including Trench, use this phrase to refer the ODE of which (1) would be the example here. Others use it to refer to the **solution** to the associated IVP.

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[2] foot note, not a square
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[2] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

The diff eqn should look like

$$x'' + \omega^2 x = 0$$

We need ω^2 : Recall $\omega^2 = \frac{k}{m}$

Since the displacement given is "in equilibrium"

we can use $\omega^2 = \frac{g}{\delta x}$

Given $\delta x = 6 \text{ in} = \frac{1}{2} \text{ ft}$, for $g = 32 \frac{\text{ft}}{\text{sec}^2}$

$$\omega^2 = \frac{g}{\delta x} = \frac{32 \text{ ft/sec}^2}{1/2 \text{ ft}} = 64 \frac{1}{\text{sec}^2}$$

The ODE is

$$x'' + 64x = 0$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

Our IVP should look like

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1$$

We need ω^2, so we need k and m.

$$W = mg \Rightarrow 4\text{ lb} = m\left(32\ \frac{\text{ft}}{\text{sec}^2}\right)$$

$$\Longrightarrow \quad m = \frac{4}{32}\text{ slugs} = \frac{1}{8}\text{ slugs}$$

From Hooke's Law $\quad W = k\,\delta x \quad \Longrightarrow \quad k = \dfrac{W}{\delta x}$

$W = 4\,\text{lb}$ and $\delta x = 6\,\text{in} = \frac{1}{2}\,\text{ft}$

so $\quad k = \dfrac{4\,\text{lb}}{\frac{1}{2}\,\text{ft}} = 8\,\dfrac{\text{lb}}{\text{ft}}$

So $\quad mx'' + kx = 0 \quad \Longrightarrow \quad \frac{1}{8}x'' + 8x = 0$

$$\Longrightarrow \quad x'' + 64x = 0$$

$x'' + 64x = 0$, $x(0) = 4$ $x'(0) = -24$

(downward is negative)

Charc. Eqn

$r^2 + 64 = 0 \Rightarrow r^2 = -64$

$r = \pm 8i$

$\alpha \pm i\beta$

$\alpha = 0$, $\beta = 8$

$x_1 = e^{0t}\cos(8t)$ $x_2 = e^{0t}\sin(8t)$

The general soln is $x = c_1 \cos(8t) + c_2 \sin(8t)$

$$x'(t) = -8c_1 \sin(8t) + 8c_2 \cos(8t)$$

$$x(0) = 4 = c_1 \cos 0 + c_2 \sin 0 = c_1 \Rightarrow c_1 = 4$$

$$x'(0) = -8c_1 \sin 0 + 8c_2 \cos(0) = -24$$

$$8c_2 = -24 \Rightarrow c_2 = -3$$

Hence the displacement

$$\boxed{x = 4\cos(8t) - 3\sin(8t)}$$