March 27 Math 2306 sec. 57 Spring 2018

Section 15: Shift Theorems

Theorem (translation in s) Suppose $\mathscr{L}\left\{f(t)\right\} = F(s)$. Then for any real number a
$$\mathscr{L}\left\{e^{at}f(t)\right\} = F(s-a).$$

For example,

$$\mathscr{L}\left\{t^n\right\} = \frac{n!}{s^{n+1}} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}t^n\right\} = \frac{n!}{(s-a)^{n+1}}.$$

$$\mathscr{L}\left\{\cos(kt)\right\} = \frac{s}{s^2+k^2} \quad \Longrightarrow \quad \mathscr{L}\left\{e^{at}\cos(kt)\right\} = \frac{s-a}{(s-a)^2+k^2}.$$

Example

(a) $\mathscr{L}\left\{t^5 e^{-3t}\right\}$

$$= \frac{5!}{(s+3)^6}$$

$$\mathscr{L}\{t^5\} = \frac{5!}{s^6}$$

Here $a = -3$

so $s - a = s + 3$

Example

(b) $\mathscr{L}^{-1}\left\{\frac{1}{s^2-4s+8}\right\}$

s^2-4s+8 is irreducible

we can complete the square

$$s^2-4s+8 = s^2-4s+4+4$$

$$= (s-2)^2+2^2$$

$$\frac{1}{s^2-4s+8} = \frac{1}{(s-2)^2+2^2}$$

$$= \frac{1}{2}\,\frac{2}{(s-2)^2+2^2}$$

Looks most like

$$\frac{k}{s^2+k^2}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2-4s+8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\ \frac{2}{(s-2)^2+2^2}\right\}$$

$$= \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2+2^2}\right\}$$

$$= \frac{1}{2}e^{2t}\sin(2t)$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathscr{U}(t-a)$ is defined by

$$\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

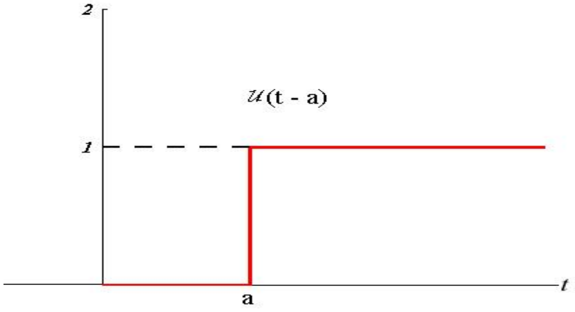

Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \le t < a \\ h(t), & t \ge a \end{cases} \quad = \; g(t) - g(t)\mathscr{U}(t-a) + h(t)\mathscr{U}(t-a)$$

We verified this last time by showing that the expression on the far right is equal to f on the two intervals $0 \le t < a$ and $t \ge a$.

Piecewise Defined Functions in Terms of $\mathscr{U}$

Write f on one line in terms of $\mathscr{U}$ as needed

$$f(t) = \begin{cases} e^t, & 0 \le t < 2 \\ t^2, & 2 \le t < 5 \\ 2t & t \ge 5 \end{cases}$$

$$f(t) = \underset{\text{on}}{e^t} - \underset{\text{off}}{e^t \mathscr{U}(t-2)} + \underset{\text{on}}{t^2 \mathscr{U}(t-2)} - \underset{\text{off}}{t^2 \mathscr{U}(t-5)} + \underset{\text{on}}{2t\, \mathscr{U}(t-5)}$$

We can verify:

For $0 \le t < 2$, $\mathscr{U}(t-2) = 0$ $\quad \mathscr{U}(t-5) = 0$

Then

$$f(t) = e^t - e^t \cdot 0 + t^2 \cdot 0 - t^2 \cdot 0 + 2t \cdot 0 = e^t$$

For $2 \le t < 5$, $\quad u(t-2) = 1 \qquad u(t-5) = 0$

Then

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 0 + 2t \cdot 0 = t^2$$

For $t \geq 5$, $\quad u(t-2) = 1 \qquad u(t-5) = 1$

Then

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 1 + 2t \cdot 1 = 2t$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t-a)\mathscr{U}(t-a) = \begin{cases} 0, & 0 \leq t < a \\ f(t-a), & t \geq a \end{cases}.$$

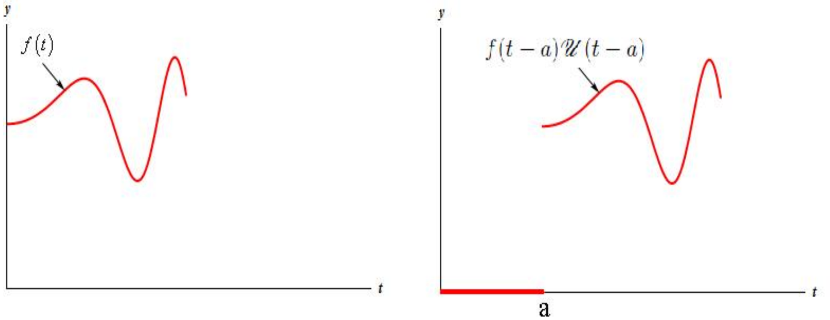

Figure: The function $f(t-a)\mathscr{U}(t-a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a.

Theorem (translation in t)

If $F(s)=\mathscr{L}\{f(t)\}$ and $a>0$, then

$$\mathscr{L}\{f(t-a)\mathscr{U}(t-a)\}=e^{-as}F(s).$$

In particular,

$$\mathscr{L}\{\mathscr{U}(t-a)\}=\frac{e^{-as}}{s}.$$

As another example,

$$\mathscr{L}\{t^n\}=\frac{n!}{s^{n+1}}\quad\Longrightarrow\quad\mathscr{L}\{(t-a)^n\mathscr{U}(t-a)\}=\frac{n!e^{-as}}{s^{n+1}}.$$

Find $\mathscr{L}\{\mathscr{U}(t-a)\}$

$$\mathcal{U}(t-a) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \geq a \end{cases}$$

$$\mathcal{L}\{\mathcal{U}(t-a)\} = \int_0^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}\,\mathcal{U}(t-a)\,dt + \int_a^\infty e^{-st}\,\mathcal{U}(t-a)\,dt$$

$$= \int_0^a e^{-st}\cdot 0\,dt + \int_a^\infty e^{-st}\cdot 1\,dt$$

$$= \int_a^\infty e^{-st}\,dt \qquad (\text{diverges if } s=0)$$

$$= \frac{-1}{s} e^{-st}\Big|_a^\infty \qquad \text{requires } s>0$$

$$= \frac{-1}{s}\left(0 - e^{-as}\right) = \frac{e^{-as}}{s}$$

Example

Find the Laplace transform $\mathscr{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$f(t) = 1 - 1\,\mathcal{U}(t-1) + t\,\mathcal{U}(t-1)$$

Let's simplify

$$= 1 - \mathcal{U}(t-1)\,(1-t)$$

$$= 1 + (t-1)\,\mathcal{U}(t-1)$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t-1)\mathscr{U}(t-1)$ to find $\mathscr{L}\{f\}$.

$$\mathscr{L}\{f(t)\} = \mathscr{L}\{1 + (t-1)\mathscr{U}(t-1)\}$$

$$= \mathscr{L}\{1\} + \mathscr{L}\{(t-1)\mathscr{U}(t-1)\}$$

$$= \frac{1}{s} + \frac{1}{s^2}e^{-s}$$

$*$ If $g(t) = t$ then $g(t-1) = t-1$ and $\mathscr{L}\{g(t)\} = \frac{1}{s^2}$

A Couple of Useful Results

Another formulation of this translation theorem is

$$\text{(1)} \quad \mathscr{L}\{g(t)\mathscr{U}(t-a)\} = e^{-as}\mathscr{L}\{g(t+a)\}.$$

Because $g(t) = g((t+a)-a)$

Example: Find $\mathscr{L}\left\{\cos t\, \mathscr{U}\left(t - \frac{\pi}{2}\right)\right\}$ $= e^{-\frac{\pi}{2}s} \mathscr{L}\left\{\cos\left(t+\frac{\pi}{2}\right)\right\}$

$$* \quad \cos\left(t+\frac{\pi}{2}\right) = \cos t \cos\frac{\pi}{2} - \sin t \sin\frac{\pi}{2}$$

$$= \cos t \cdot 0 - \sin t \cdot 1 = -\sin t$$

$$\mathcal{L}\{\cos t\,\mathcal{U}(t-\pi/2)\} = e^{-\frac{\pi}{2}s}\mathcal{L}\{\cos(t+\tfrac{\pi}{2})\}$$

$$= e^{-\frac{\pi}{2}s}\mathcal{L}\{-\sin t\}$$

$$= e^{-\frac{\pi}{2}s}\left(\frac{-1}{s^2+1^2}\right)$$

$$= \frac{-e^{-\frac{\pi}{2}s}}{s^2+1}$$

A Couple of Useful Results

The inverse form of this translation theorem is

$$(2) \quad \mathscr{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\mathscr{U}(t-a).$$

Example: Find $\mathscr{L}^{-1}\left\{\dfrac{e^{-2s}}{s(s+1)}\right\}$ Here $F(s) = \dfrac{1}{s(s+1)}$

Do decomp on $F(s)$

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \implies 1 = A(s+1) + Bs$$

$s=0 \quad 1 = A$

$s=-1 \quad 1 = -B \implies B = -1$

$F(s) = \dfrac{1}{s} - \dfrac{1}{s+1}$ and

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= \underbrace{1 - e^{-t}}_{\text{this is our } f(t)}$$

for $f(t-a)\mathcal{U}(t-a)$

$$\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\} = \left(1 - e^{-(t-2)}\right)\mathcal{U}(t-2)$$