

Section 15: Shift Theorems

Theorem (translation in s) Suppose $\mathcal{L}\{f(t)\} = F(s)$. Then for any real number a

$$\mathcal{L}\{e^{at}f(t)\} = F(s - a).$$

For example,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathcal{L}\{e^{at}t^n\} = \frac{n!}{(s - a)^{n+1}}.$$

$$\mathcal{L}\{\cos(kt)\} = \frac{s}{s^2 + k^2} \implies \mathcal{L}\{e^{at}\cos(kt)\} = \frac{s - a}{(s - a)^2 + k^2}.$$

Example

(a) $\mathcal{L} \{ t^5 e^{-3t} \}$

$$\mathcal{L} \{ t^5 \} = \frac{5!}{s^6}$$

$$= \frac{5!}{(s+3)^6}$$

for $a = -3$,
 $s-a = s+3$

Example

$$(b) \quad \mathcal{L}^{-1} \left\{ \frac{1}{s^2 - 4s + 8} \right\}$$

$s^2 - 4s + 8$ is irreducible
we complete the square

$$\begin{aligned}s^2 - 4s + 8 &= s^2 - 4s + 4 + 4 \\&= (s-2)^2 + 4\end{aligned}$$

Lodas like

$$\begin{aligned}\frac{1}{s^2 - 4s + 8} &= \frac{1}{(s-2)^2 + 2^2} \\&= \frac{1}{2} \cdot \frac{2}{(s-2)^2 + 2^2}\end{aligned}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2 - 4s + 8}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2} - \frac{2}{(s-2)^2 + 2^2}\right\}$$

$$= \frac{1}{2} \mathcal{L}^{-1}\left\{\frac{2}{(s-2)^2 + 2^2}\right\}$$

$$= \frac{1}{2} e^{zt} \sin(zt)$$

The Unit Step Function

Let $a \geq 0$. The unit step function $\mathcal{U}(t - a)$ is defined by

$$\mathcal{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

another notation $\mathcal{U}(t-a) = u_a(t)$

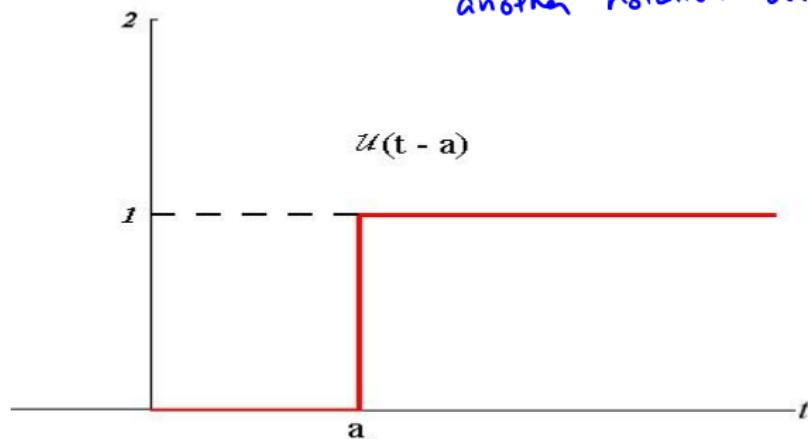


Figure: We can use the unit step function to provide convenient expressions for piecewise defined functions.

Piecewise Defined Functions

Verify that

$$f(t) = \begin{cases} g(t), & 0 \leq t < a \\ h(t), & t \geq a \end{cases} = g(t) - g(t)U(t-a) + h(t)U(t-a)$$

Consider $0 \leq t < a$, $U(t-a) = 0$

$$f(t) = g(t) - g(t) \cdot 0 + h(t) \cdot 0 = g(t)$$

For $a \leq t$, $U(t-a) = 1$

$$f(t) = g(t) - g(t) \cdot 1 + h(t) \cdot 1 = h(t) \quad \text{as required}$$

Piecewise Defined Functions in Terms of \mathcal{U}

Write f on one line in terms of \mathcal{U} as needed

$$f(t) = \begin{cases} e^t, & 0 \leq t < 2 \\ t^2, & 2 \leq t < 5 \\ 2t & t \geq 5 \end{cases}$$

$$f(t) = e^t - e^t u(t-2) + t^2 u(t-2) - t^2 u(t-5) + 2t u(t-5)$$



Let's verify : for $0 \leq t < 2$ $u(t-2) = 0$ $u(t-5) = 0$

$$f(t) = e^t - e^t \cdot 0 + t^2 \cdot 0 - t^2 \cdot 0 + 2t \cdot 0 = e^t$$

For $2 \leq t < 5$, $u(t-2) = 1$ $u(t-5) = 0$

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 0 + 2t \cdot 0 = t^2$$

For $t \geq 5$, $u(t-2) = 1$ $u(t-5) = 1$

$$f(t) = e^t - e^t \cdot 1 + t^2 \cdot 1 - t^2 \cdot 1 + 2t \cdot 1 = 2t$$

Translation in t

Given a function $f(t)$ for $t \geq 0$, and a number $a > 0$

$$f(t - a)\mathcal{U}(t - a) = \begin{cases} 0, & 0 \leq t < a \\ f(t - a), & t \geq a \end{cases} .$$

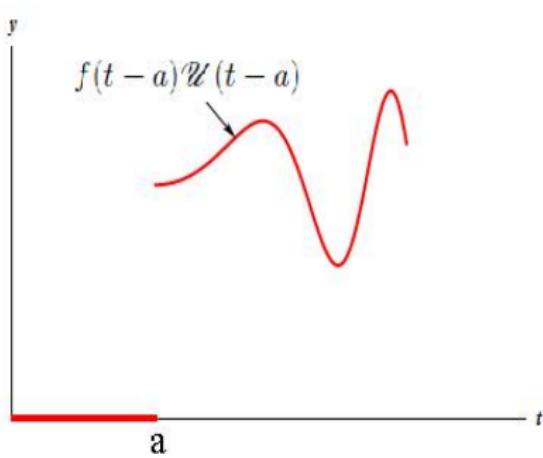
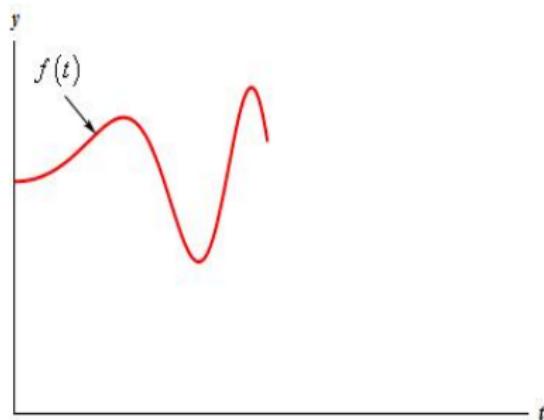


Figure: The function $f(t - a)\mathcal{U}(t - a)$ has the graph of f shifted a units to the right with value of zero for t to the left of a .

Theorem (translation in t)

If $F(s) = \mathcal{L}\{f(t)\}$ and $a > 0$, then

$$\mathcal{L}\{f(t - a)\mathcal{U}(t - a)\} = e^{-as}F(s).$$

In particular,

$$\mathcal{L}\{\mathcal{U}(t - a)\} = \frac{e^{-as}}{s}.$$

As another example,

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}} \implies \mathcal{L}\{(t - a)^n\mathcal{U}(t - a)\} = \frac{n!e^{-as}}{s^{n+1}}.$$

Find $\mathcal{L}\{\mathcal{U}(t-a)\}$

$$u(t-a) = \begin{cases} 0, & 0 \leq t < a \\ 1, & t \geq a \end{cases}$$

By definition

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st} u(t-a) dt + \int_a^\infty e^{-st} u(t-a) dt$$

$$= \int_0^a e^{-st} \cdot 0 dt + \int_a^\infty e^{-st} \cdot 1 dt$$

$$= \int_a^\infty e^{-st} dt \quad \text{Diverges if } s=0$$

$$= \frac{-1}{s} e^{-st} \Big|_a^\infty$$

Converges if
 $s > 0$

$$= \frac{-1}{s} (0 - e^{-as})$$

$$= \frac{e^{-as}}{s} \quad \text{for } s > 0$$

Example

Find the Laplace transform $\mathcal{L}\{f(t)\}$ where

$$f(t) = \begin{cases} 1, & 0 \leq t < 1 \\ t, & t \geq 1 \end{cases}$$

(a) First write f in terms of unit step functions.

$$\begin{aligned} f(t) &= 1 - 1u(t-1) + tu(t-1) \\ &= 1 + (-1 + t)u(t-1) \\ &= 1 + (t-1)u(t-1) \end{aligned}$$

Example Continued...

(b) Now use the fact that $f(t) = 1 + (t - 1)U(t - 1)$ to find $\mathcal{L}\{f\}$.

$$\begin{aligned} \mathcal{L}\{f(t)\} &= \mathcal{L}\{1 + (t-1)U(t-1)\} \\ &= \mathcal{L}\{1\} + \mathcal{L}\{(t-1)U(t-1)\} \\ &= \frac{1}{s} + \frac{1}{s^2} \cdot e^{-s} = \frac{1}{s} + \frac{e^{-s}}{s^2} \end{aligned}$$

* If $g(t) = t$, then $g(t-1) = t-1$ $\mathcal{L}\{t\} = \frac{1}{s^2}$

A Couple of Useful Results

Another formulation of this translation theorem is

$$(1) \quad \mathcal{L}\{g(t)\mathcal{U}(t-a)\} = e^{-as}\mathcal{L}\{g(t+a)\}.$$

Because $g(t) = g((t+a) - a)$

Example: Find $\mathcal{L}\{\cos t \mathcal{U}\left(t - \frac{\pi}{2}\right)\} = e^{-\frac{\pi}{2}s} \mathcal{L}\{\cos(t + \frac{\pi}{2})\}$

$$\begin{aligned}\cos(t + \frac{\pi}{2}) &= \cos t \cos \frac{\pi}{2} - \sin t \sin \frac{\pi}{2} \\ &= \cos t \cdot 0 - \sin t \cdot 1 = -\sin t\end{aligned}$$

$$y\{ \cos t u(t-\pi/2) \} = e^{-\frac{\pi}{2}s} f\{-\sin t\}$$

$$= -e^{-\frac{\pi}{2}s} \left(\frac{1}{s^2 + 1^2} \right)$$

$$= \frac{-e^{-\frac{\pi}{2}s}}{s^2 + 1}$$

A Couple of Useful Results

The inverse form of this translation theorem is

$$(2) \quad \mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)U(t-a).$$

Example: Find $\mathcal{L}^{-1}\left\{\frac{e^{-2s}}{s(s+1)}\right\}$ Here $F(s) = \frac{1}{s(s+1)}$

Do partial fraction decomp

$$\frac{1}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1}$$

$$s=0 \quad 1=A$$

$$\Rightarrow 1 = A(s+1) + Bs$$

$$s=-1 \quad 1 = -B$$

$$F(s) = \frac{1}{s} - \frac{1}{s+1}$$

$$\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\}$$

$$= 1 - e^{-t} = f(t)$$

$$\mathcal{L}^{-1}\left\{e^{-2s} \left(\frac{1}{s(s+1)}\right)\right\} = (1 - e^{-(t-2)})u(t-2)$$