March 27 Math 3260 sec. 55 Spring 2018

Section 4.6: Rank

Definition: The **row space**, denoted Row A, of an $m \times n$ matrix A is the subspace of $\mathbb{R}^n$ spanned by the rows of A.

We now have three vector spaces associated with an $m \times n$ matrix A, its column space, null space, and row space.

Theorem

If two matrices A and B are row equivalent, then their row spaces are the same.

In particular, if B is an echelon form of the matrix A, then the nonzero rows of B form a basis for Row B—and also for Row A since these are the same space.

Example

A matrix A along with its rref is shown.

$$A = \begin{bmatrix} -2 & -5 & 8 & 0 & -17 \\ 1 & 3 & -5 & 1 & 5 \\ 3 & 11 & -19 & 7 & 1 \\ 1 & 7 & -13 & 5 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -2 & 0 & 3 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(a) Find a basis for Row A and state the dimension dim Row A.

We can use the non zero rows of the rref

A basis for Row A is $\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -5 \end{bmatrix} \right\}$

$\dim(\text{Row } A) = 3$

Example continued ...

(b) Find a basis for Col A and state its dimension.

We can use the pivot columns of A which are columns 1, 2, and 4.

A basis for Col A is $\left\{ \begin{bmatrix} -2 \\ 1 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ 3 \\ 11 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 7 \\ 5 \end{bmatrix} \right\}$

$\dim(\text{Col} A) = 3$

Example continued ...

(c) Find a basis for Nul A and state its dimension.

From the rref, if $A\vec{x}=\vec{0}$ then

$$x_1 = -x_3 - x_5$$
$$x_2 = 2x_3 - 3x_5$$
$$x_4 = 5x_5$$
$$x_3, x_5 \text{ - free}$$

So

$$\vec{x} = \begin{bmatrix} -x_3 - x_5 \\ 2x_3 - 3x_5 \\ x_3 \\ 5x_5 \\ x_5 \end{bmatrix} = x_3 \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix}$$

A basis for Nul A is $\left\{ \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -3 \\ 0 \\ 5 \\ 1 \end{bmatrix} \right\}$.

$\dim(\text{Nul} A) = 2$

Remarks

- We can naturally associate three vector spaces with an $m \times n$ matrix A. Row A and Nul A are subspaces of $\mathbb{R}^n$ and Col A is a subspace of $\mathbb{R}^m$.

- Careful! The rows of the rref do span Row A. But we go back to the columns in the original matrix to get vectors that span Col A. (Get a basis for Col A from A itself!)

- Careful Again! Just because the first three rows of the rref span Row A **does not mean** the first three rows of A span Row A. (Get a basis for Row A from the rref!)

Remarks

- Row operations preserve row space, but change linear dependence relations of rows. Row operations change column space, but preserve linear dependence relations of columns.

- Another way to obtain a basis for Row A is to take the transpose A^T and do row operations. We have the following relationships:

$$\text{Col } A \;=\; \text{Row } A^T \quad \text{and} \quad \text{Row } A \;=\; \text{Col } A^T.$$

- The dimension of the null space is called the **nullity**.

Rank

Definition: The **rank** of a matrix A (denoted rank A) is the dimension of the column space of A.

Theorem: For $m \times n$ matrix A, dim Col A = dim Row A = rank A. Moreover

$$\text{rank } A + \dim \text{Nul } A = n.$$

Note: This theorem states the rather obvious fact that

$$\left\{ \begin{array}{c} \text{number of} \\ \text{pivot columns} \end{array} \right\} + \left\{ \begin{array}{c} \text{number of} \\ \text{non-pivot columns} \end{array} \right\} = \left\{ \begin{array}{c} \text{total number} \\ \text{of columns} \end{array} \right\}.$$

Examples

(1) A is a 5×4 matrix with rank $A = 4$. What is dim Nul A?

$$\text{rank } A + \dim(\text{Nul} A) = n = 4$$

$$4 + \dim(\text{Nul} A) = 4 \Rightarrow \dim(\text{Nul} A) = 0$$

(2) If A is 7×5 and dim Col $A = 2$. Determine the nullity[1] of A and rank A^T.

$$\text{rank } A + \text{nullity} = n = 5 \qquad \text{rank } A = \dim \text{Col } A = 2$$

$$2 + \text{nullity} = 5$$

$$\Rightarrow \text{nullity} = 3$$

$$\text{rank } A^T = \dim \text{Col } A^T = \dim \text{Row } A = \dim \text{Col } A = 2$$

Addendum to Invertible Matrix Theorem

Let A be an $n \times n$ matrix. The following are equivalent to the statement that A is invertible.

(m) The columns of A form a basis for $\mathbb{R}^n$

(n) $\text{Col } A = \mathbb{R}^n$

(o) $\dim \text{Col } A = n$

(p) $\text{rank } A = n$

(q) $\text{Nul } A = \{\mathbf{0}\}$

(r) $\dim \text{Nul } A = 0$

Section 6.1: Inner Product, Length, and Orthogonality

Recall: A vector **u** in $\mathbb{R}^n$ can be considered an $n \times 1$ matrix. It follows that $\mathbf{u}^T$ is a $1 \times n$ matrix

$$\mathbf{u}^T = [u_1\ u_2\ \cdots\ u_n].$$

Definition: For vectors **u** and **v** in $\mathbb{R}^n$ we define the **inner product** of **u** and **v** (also called the **dot product**) by the **matrix product**

$$\mathbf{u}^T\mathbf{v} = \underbrace{[u_1\ u_2\ \cdots\ u_n]}_{1\times n} \underbrace{\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}}_{n\times 1} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

Note that this product produces a scalar. It is sometimes called a *scalar product*.

Theorem (Properties of the Inner Product)

We'll use the notation $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

often written as $\langle \vec{u}, \vec{v} \rangle$

Theorem: For $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and real scalar c

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(c) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$

(d) $\mathbf{u} \cdot \mathbf{u} \geq 0$, with $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

The Norm

The property $\mathbf{u} \cdot \mathbf{u} \geq 0$ means that $\sqrt{\mathbf{u} \cdot \mathbf{u}}$ always exists as a real number.

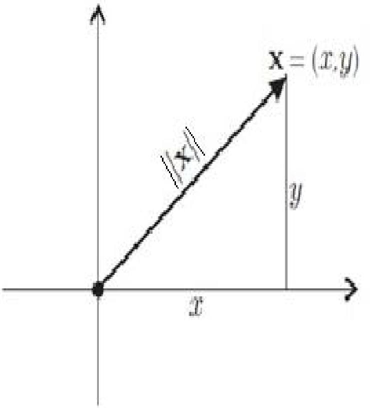
Definition: The **norm** of the vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative number denoted and defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

where $v_1, v_2, \ldots, v_n$ are the components of $\mathbf{v}$.

As a directed line segment, the norm is the same as the **length**.

Norm and Length



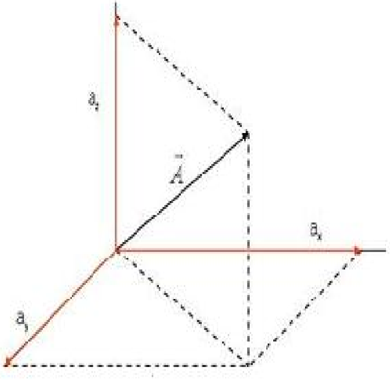


Figure: In $\mathbb{R}^2$ or $\mathbb{R}^3$, the norm corresponds to the classic geometric property of length.

Unit Vectors and Normalizing

Theorem: For vector $\mathbf{v}$ in $\mathbb{R}^n$ and scalar c

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$

Definition: A vector $\mathbf{u}$ in $\mathbb{R}^n$ for which $\|\mathbf{u}\| = 1$ is called a **unit vector.**

Remark: Given any nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, we can obtain a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

This process, of dividing out the norm, is called **normalizing** the vector $\mathbf{v}$.

Example

Show that $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector.

Recall $\|\vec{v}\|$ is a scalar and $\|\vec{v}\| > 0$ for nonzero $\vec{v}$.

$$\left\| \frac{\vec{v}}{\|\vec{v}\|} \right\| = \left\| \frac{1}{\|\vec{v}\|} \vec{v} \right\| = \left| \underset{\text{positive}}{\frac{1}{\|\vec{v}\|}} \right| \|\vec{v}\|$$

$$= \frac{1}{\|\vec{v}\|} \|\vec{v}\| = \frac{\|\vec{v}\|}{\|\vec{v}\|} = 1$$

Example

Find a unit vector in the direction of $\mathbf{v} = (1, 3, 2)$.

$$\|\vec{v}\| = \sqrt{1^2+3^2+2^2} = \sqrt{14}$$

If we call the unit vector $\vec{u}$,

$$\vec{u} = \frac{1}{\|\vec{v}\|}\vec{v} = \frac{1}{\sqrt{14}}\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{14}} \\ \frac{3}{\sqrt{14}} \\ \frac{2}{\sqrt{14}} \end{bmatrix}$$

Distance in $\mathbb{R}^n$

Definition: For vectors **u** and **v** in $\mathbb{R}^n$, the **distance between u and v** is denoted and defined by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Example: Find the distance between $\mathbf{u} = (4, 0, -1, 1)$ and $\mathbf{v} = (0, 0, 2, 7)$.

$$\vec{u} - \vec{v} = (4, 0, -3, -6)$$

$$\text{dist}(\vec{u}, \vec{v}) = \sqrt{4^2 + 0^2 + (-3)^2 + (-6)^2} = \sqrt{16 + 9 + 36} = \sqrt{61}$$

Orthogonality

Definition: Two vectors are $\mathbf{u}$ and $\mathbf{v}$ **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

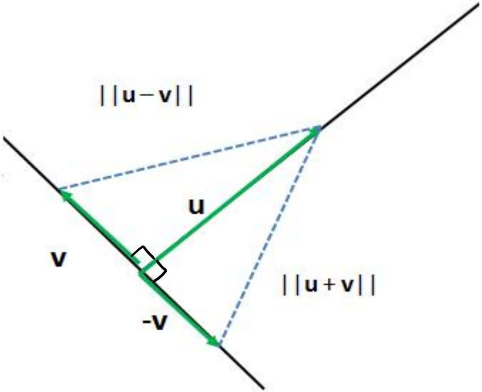


Figure: Note that two vectors are perpendicular if $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$

Orthogonal and Perpendicular

Show that $\|\mathbf{u}-\mathbf{v}\| = \|\mathbf{u}+\mathbf{v}\|$ if and only if $\mathbf{u}\cdot\mathbf{v} = 0$.

Note

$$\|\vec{u}-\vec{v}\|^2 = (\vec{u}-\vec{v})\cdot(\vec{u}-\vec{v})$$

$$= \vec{u}\cdot\vec{u} - \vec{v}\cdot\vec{u} - \vec{u}\cdot\vec{v} + \vec{v}\cdot\vec{v}$$

$$= \|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

Similarly

$$\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

If $\|\vec{u}-\vec{v}\| = \|\vec{u}+\vec{v}\|$, then $\|\vec{u}-\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$

So
$$\|\vec{u}\|^2 - 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

$$\Rightarrow \quad -2\vec{u}\cdot\vec{v} = 2\vec{u}\cdot\vec{v}$$

$$\Rightarrow \quad 0 = 4\vec{u}\cdot\vec{v} \qquad \text{Hence} \quad \vec{u}\cdot\vec{v} = 0.$$

Conversely, if $\vec{u}\cdot\vec{v}=0$, then

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 = \|\vec{u}+\vec{v}\|^2$$

Since these are nonnegative

$$\|\vec{u}-\vec{v}\| = \|\vec{u}+\vec{v}\|.$$

The Pythagorean Theorem

Theorem: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

From before, $\|\vec{u}+\vec{v}\|^2 = \|\vec{u}\|^2 + 2\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$.

The result follows directly.

Orthogonal Complement

Definition: Let W be a subspace of $\mathbb{R}^n$. A vector $\mathbf{z}$ in $\mathbb{R}^n$ is said to be **orthogonal to** W if $\mathbf{z}$ is orthogonal to every vector in W.

i.e. $\vec{z} \cdot \vec{w} = 0$ for every $\vec{w}$ in W

Definition: Given a subspace W of $\mathbb{R}^n$, the set of all vectors orthogonal to W is called the **orthogonal complement** of W and is denoted by

$$W^{\perp}.$$

reads as "W perp"