# March 28 Math 1190 sec. 62 Spring 2017

## Section 4.8: Antiderivatives; Differential Equations

**Definition:** A function $F$ is called an antiderivative of $f$ on an interval $I$ if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

As a consequence of the Mean Value Theorem, we have ...

**Theorem:** If $F$ is any antiderivative of $f$ on an interval $I$, then the *most general* antiderivative of $f$ on $I$ is

$$F(x) + C \quad \text{where } C \text{ is an arbitrary constant.}$$

# Find the most general antiderivative of $f$.

$$f(x) = \frac{4x^4+1}{x}, \quad I = (0,\infty)$$

we want to recognize f as a derivative.

we can use a little algebra to make it recognizable,

$$f(x) = \frac{4x^4+1}{x} = \frac{4x^4}{x} + \frac{1}{x} = 4x^3 + \frac{1}{x}$$

$$F(x) = x^4 + \ln x + C$$ for arbitrary constant $C$

Check $F'(x) = 4x^3 + \frac{1}{x} + 0 = 4x^3 + \frac{1}{x}$ ✓

# Question: Find the most general antiderivative of $f$.

$$f(x) = \frac{3x-1}{x}, \quad I = (0, \infty)$$

$$f(x) = \frac{3x}{x} - \frac{1}{x} = 3 - \frac{1}{x}$$

(a) $F(x) = \dfrac{\frac{3x^2}{2} - x}{\frac{x^2}{2}} + C$

we can't take antiderivatives of individual factors in a product or quotient

(b) $F(x) = \dfrac{1}{x^2} + C$

(c) $F(x) = 3x - \ln x + C$

# Find the most general antiderivative of

$$f(x) = x^n, \quad \text{where } n = 1, 2, 3, \ldots$$

We found that $F(x) = \frac{1}{n+1}x^{n+1} + C$. This is called the **power rule** for antiderivatives. It holds for all real numbers $n$ except for $n = -1$. For example

$$\text{If} \quad f(x) = x^{1/2} \quad \text{then} \quad F(x) = \frac{1}{3/2}x^{3/2} + C = \frac{2}{3}x^{3/2} + C.$$

here $n = \frac{1}{2}$ and $n+1 = \frac{1}{2}+1 = \frac{3}{2}$

# Some general results[1]:

(See the table on page 330 in Sullivan & Miranda for a more comprehensive list.)

| Function | Particular Antiderivative | Function | Particular Antiderivative |
|---|---|---|---|
| $cf(x)$ | $cF(x)$ | $\cos x$ | $\sin x$ |
| $f(x)+g(x)$ | $F(x)+G(x)$ | $\sin x$ | $-\cos x$ |
| $x^n$, $n \neq -1$ | $\frac{x^{n+1}}{n+1}$ | $\sec^2 x$ | $\tan x$ |
| $\frac{1}{x}$ | $\ln \lvert x \rvert$ | $\csc x \cot x$ | $-\csc x$ |
| $\frac{1}{x^2+1}$ | $\tan^{-1} x$ | $\frac{1}{\sqrt{1-x^2}}$ | $\sin^{-1} x$ |

[1] We'll use the term particular antiderivative to refer to any antiderivative that has no arbitrary constant in it.

# Example

Find the most general antiderivative of $h(x) = x\sqrt{x}$ on $(0, \infty)$.

h is a power function

$h(x) = x \cdot x^{1/2} = x^{3/2}$

power rule $\frac{x^{n+1}}{n+1}$

w/ $n = \frac{3}{2}$

$\frac{3}{2} + 1 = \frac{5}{2}$

$H(x) = \frac{1}{5/2} x^{5/2} + C$

$\Rightarrow \boxed{H(x) = \frac{2}{5} x^{5/2} + C}$

# Example

Determine the function $H(x)$ that satisfies the following conditions

$$H'(x) = x\sqrt{x}, \quad \text{for all } x > 0, \text{ and } H(1) = 0.$$

the point $(1,0)$ must be on the graph of $H$.

We already know that all $H$ satisfying $H'(x) = x\sqrt{x}$ must be of the form

$$H(x) = \frac{2}{5}x^{5/2} + C$$

Now impose $H = 0$ when $x = 1$.

$$0 = \frac{2}{5}(1)^{5/2} + C \Rightarrow 0 = \frac{2}{5} + C \Rightarrow C = \frac{-2}{5}$$

The solution is $\boxed{H(x) = \frac{2}{5}x^{5/2} - \frac{2}{5}}$.

# Example

A **differential equation** is an equation that involves the derivative(s) of an unknown function. **Solving** such an equation would mean finding such an unknown function.

For example, we just solved the differential equation

$$\frac{dH}{dx} = x\sqrt{x}$$

subject to the additional condition that $H = 0$ when $x = 1$.

The condition $H(1) = 0$ is usually called an **initial condition** if $x$ represents time. It may be called a **boundary condition** if $x$ represents space.

# Question

The **most general** solution to the differential equation

$$\frac{dy}{dx} = 2x + 1 \quad \text{is}$$

(a) $y = 2x + 1 + C$

(b) $y = x^2 + 1 + C$

(c) $y = x^2 + x$

(d) $y = x^2 + x + C$

# Question

The solution to the differential equation subject to the boundary condition

$$\frac{dy}{dx} = 2x + 1 \qquad y(1) = -2$$

$y(1) = 1^2 + 1 + C = -2$

$C = -2 - 2 = -4$

(a) $y = x^2 + x - 2$

(b) $y = x^2 + x - 4$

(c) $y = x^2 + x$

(d) $y = x^2 + x + C - 2$

# Example

A particle moves along the $x$-axis so that its acceleration at time $t$ is given by

$$a(t) = 12t - 2 \quad \text{m/sec}^2.$$

At time $t = 0$, the velocity $v$ and position $s$ of the particle are known to be

$$v(0) = 3 \quad \text{m/sec, and} \quad s(0) = 4 \text{ m}.$$

Find the position $s(t)$ of the particle for all $t > 0$.

We know that $a(t) = \frac{dv}{dt}$. So we have

$\frac{dv}{dt} = 12t - 2$ with $v(0) = 3$

power rule

$\frac{x^{n+1}}{n+1}$

$v = 12 \cdot \frac{t^2}{2} - 2t + C$

$v(t) = 6t^2 - 2t + C$ impose $v(0) = 3$

$3 = 6\cdot 0^2 - 2\cdot 0 + C \implies C = 3$

So $v(t) = 6t^2 - 2t + 3$ m/sec

Note $v(t) = \frac{ds}{dt}$

So we need to solve

$\frac{ds}{dt} = 6t^2 - 2t + 3$ subject to $s(0) = 4$

$$s(t) = 6\frac{t^3}{3} - 2\frac{t^2}{2} + 3t + C$$

$$s(t) = 2t^3 - t^2 + 3t + C$$

Impose $s(0) = 4$

$$4 = 2\cdot 0^3 - 0^2 + 3\cdot 0 + C \implies C = 4$$

The position is

$$\boxed{s(t) = 2t^3 - t^2 + 3t + 4 \quad m}$$

# Section 5.1: Area (under the graph of a nonnegative function)

We will investigate the area enclosed by the graph of a function $f$. We'll make the following assumptions (for now):

- $f$ is continuous on the interval $[a, b]$, and
- $f$ is nonnegative, i.e $f(x) \geq 0,$ on $[a, b]$.

**Our Goal:** Find the area of such a region.

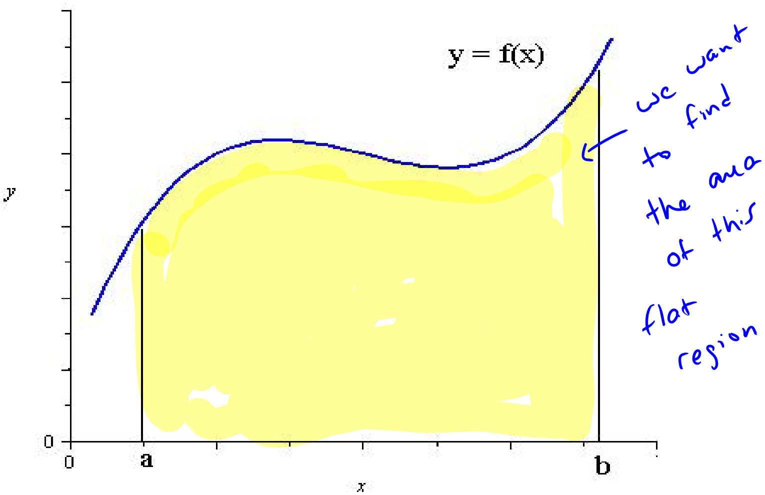
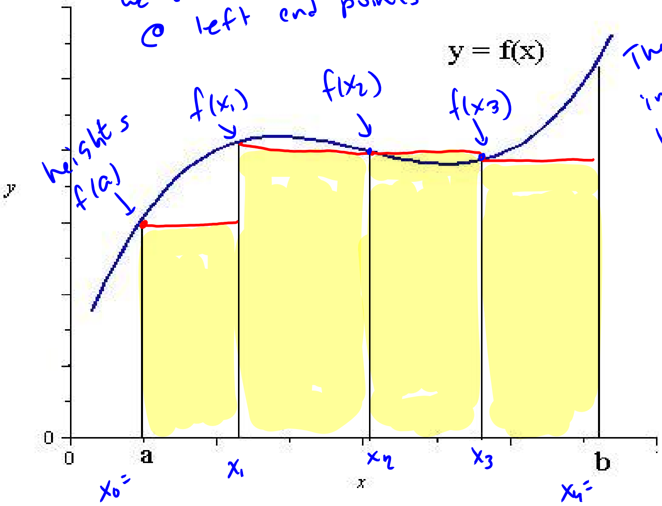



Figure: Region under a positive curve $y = f(x)$ on an interval $[a, b]$.

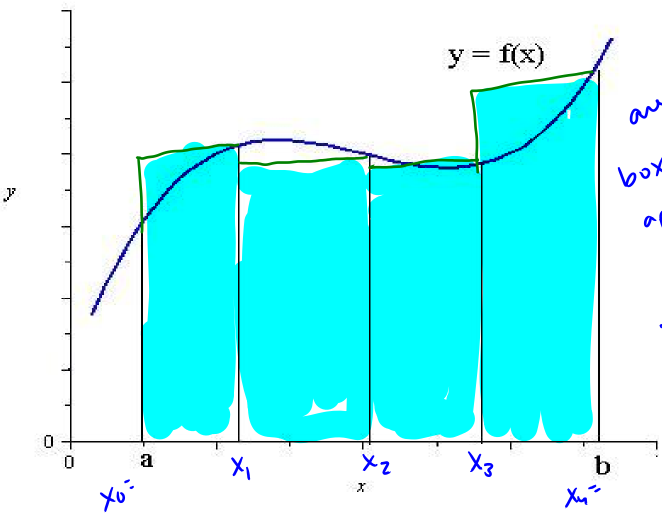
we could make rectangles using heights @ left end points




The area in the boxes approximates the true area

Figure: We could approximate the area by filling the space with rectangles.

We could use right end points instead




area in the boxes approximates the true area

Figure: We could approximate the area by filling the space with rectangles.

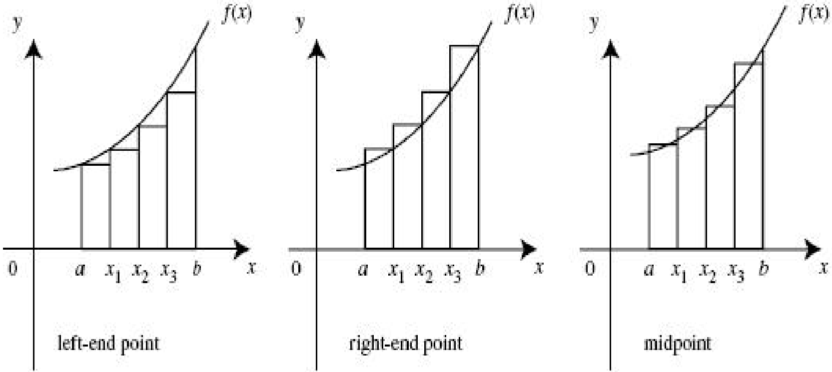



Figure: Some choices as to how to define the heights.

# Approximating Area Using Rectangles

We can experiment with

- Which points to use for the heights (left, right, middle, other....)

- How many rectangles we use

to try to get a good approximation.

**Definition:** We will define the true area to be value we obtain taking the limit as the number of rectangles goes to $+\infty$.

# Some terminology

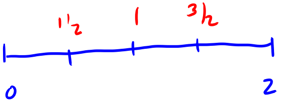
- A **Partition** $P$ of an interval $[a, b]$ is a collection of points $\{x_0, x_1, ..., x_n\}$ such that
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$
- A **Subinterval** is one of the intervals $x_{i-1} \le x \le x_i$ determined by a partition.
- The width of a subinterval is denoted $\Delta x_i = x_i - x_{i-1}$. If they are all the same size (equal spacing), then
$$\Delta x = \frac{b-a}{n},$$ and this is called the **norm** of the partition.
- A set of **sample points** is a set $\{c_1, c_2, \ldots, c_n\}$ such that $x_{i-1} \le c_i \le x_i$.

Taking the number of rectangles to $\infty$ is the same as taking the width $\Delta x \to 0$.

# Example:

Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm $\Delta x$.

(a) For $n = 4$




Norm

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$x_0 = 0$

$x_1 = \frac{1}{2}$

$x_2 = 1$

$x_3 = \frac{3}{2}$

$x_4 = 2$

Note

$x_0 = a$

$x_1 = a + 1\Delta x$

$x_2 = a + 2\Delta x$

$x_3 = a + 3\Delta x$

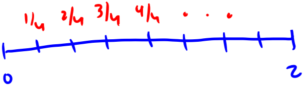
In general

$x_i = a + i\Delta x$

# Example:

Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm $\Delta x$.

(b) For $n = 8$




$$\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = \frac{1}{4}$$

$x_0 = 0$ $\quad x_4 = 1$ $\quad x_8 = 2$

$x_1 = \frac{1}{4}$ $\quad x_5 = 5/4$

$x_2 = 1/2$ $\quad x_6 = 3/2$

$x_3 = 3/4$ $\quad x_7 = 7/4$

# Question

$n=6$

Write an equally spaced partition of the interval $[0,2]$ with 6 subintervals, and determine the norm $\Delta x$.

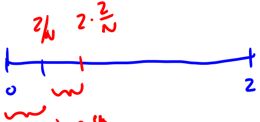
$\Delta x = \frac{b-a}{n} = \frac{2-0}{6} = \frac{1}{3}$

(a) $\left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right\}$ $\quad \Delta x = \frac{1}{3}$

(b) $\left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right\}$ $\quad \Delta x = \frac{1}{6}$

(c) $\left\{0, \frac{1}{6}, \frac{1}{3}, 1, \frac{5}{6}, \frac{7}{6}, 2\right\}$ $\quad \Delta x = \frac{1}{3}$

(c) Find an equally spaced partition of $[0, 2]$ having $N$ subintervals. What is the norm $\Delta x$?




Distance is $\Delta x = \frac{2}{N}$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{N} = \frac{2}{N}$$

$x_0 = 0$

$x_1 = \frac{2}{N}$

$x_2 = 2 \cdot \frac{2}{N} = \frac{4}{N}$

$x_3 = 3 \cdot \frac{2}{N} = \frac{6}{N}$

$x_i = i\frac{2}{N}$

for $i = 1, 2, 3, \ldots, N$

Note $x_N = N\frac{2}{N} = 2$ as expected

Note: $x_i = a + i\Delta x$ where $a = 0$ and $\Delta x = \frac{2}{N}$

# Approximating area with a Partition and sample points

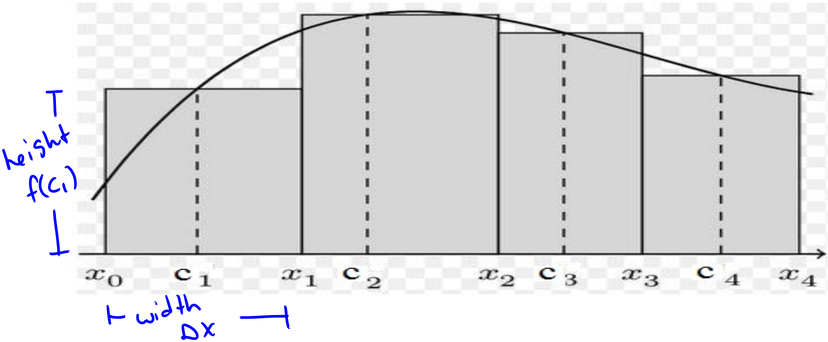



Figure: Area = $f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x$. This can be written as

$$\sum_{i=1}^{n} f(c_i)\Delta x.$$ $\leftarrow$ called a Riemann Sum

# Sum Notation

$\sum$ is the capital letter *sigma*, basically a capital Greek "S".

If $a_1, a_2, \dots, a_n$ are a collection of real numbers, then

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

This is read as

*the sum from i equals* 1 *to n of* $a_i$ *(a sub i).*

For example

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

$$\sum_{i=1}^{3} 2i^2 = 2\cdot 1^2 + 2\cdot 2^2 + 2\cdot 3^2 = 2 + 8 + 18 = 28$$

In general, an equally spaced partition of $[a, b]$ with $n$ subintervals means

- $\Delta x = \frac{b-a}{n}$
- $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, i.e. $x_i = a + i\Delta x$
- Taking heights to be

height width

$$\text{left ends} \quad c_i = x_{i-1} \quad \text{area} \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$\text{right ends} \quad c_i = x_i \quad \text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

- The true area exists (for $f$ continuous) and is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

# Lower and Upper Sums

The standard way to set up these sums is to take $c_i$ such that

$$f(c_i) \quad \text{is the abs. minimum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set $A_L$

$$A_L = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

This is called a **Lower Riemann sum**.

# Lower and Upper Sums

Then, we take $C_i$ such that

$$f(C_i) \quad \text{is the abs. maximum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set $A_U$

$$A_U = \lim_{n\to\infty} \sum_{i=1}^{n} f(C_i)\Delta x.$$

This is called a **Upper Riemann sum**.

# Lower and Upper Sums

If $f$ is continuous on $[a, b]$, then it will necessarily be that

$$A_L = A_U.$$

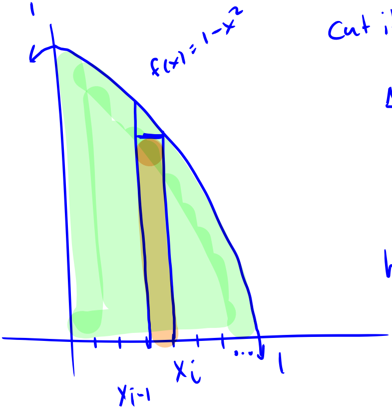
This value is the true area.

In practice, these are tough to compute unless $f$ is only increasing or only decreasing. So instead, we tend to use left and right sums.

# Example: Find the area under the curve $f(x) = 1 - x^2$, $0 \le x \le 1$.

Use right end points $c_i = x_i$ and assume the following identity

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6} \quad \text{(sum of first } n \text{ squares)}$$




Cut it into $n$ equal pieces

$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$ ← our width

$x_0 = 0, \quad x_i = 0 + i\Delta x = i\frac{1}{n} = \frac{i}{n}$

height is $f(x_i) = 1 - (x_i)^2$

One rectangle has area $f(x_i)\Delta x$

$$f(x_i)\,\Delta x = (1-(x_i)^2)\cdot \Delta x = \left(1-\left(\frac{i}{n}\right)^2\right)\cdot \frac{1}{n}$$

$$\text{area} \approx \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n}\left(1-\frac{i^2}{n^2}\right)\frac{1}{n}$$

$$= \sum_{i=1}^{n}\left(\frac{1}{n}-\frac{i^2}{n^3}\right)$$

$$= \sum_{i=1}^{n}\frac{1}{n} - \sum_{i=1}^{n}\frac{i^2}{n^3}$$

Note $\sum_{i=1}^{n} \frac{1}{n} = \underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \dots + \frac{1}{n}}_{n \text{ times}} = n\left(\frac{1}{n}\right) = 1$

and $\sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{n^3}\left(1^2 + 2^2 + \dots + n^2\right) = \frac{1}{n^3} \frac{2n^3 + 3n^2 + n}{6}$

So we have

$$\text{area} \approx 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$

To get the true area, take $n \to \infty$

$$\text{area} = \lim_{n\to\infty}\left(1 - \frac{2n^3+3n^2+n}{6n^3}\right)$$

$$= \lim_{n\to\infty}\left(1 - \frac{2n^3+3n^2+n}{6n^3}\left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right)\right)$$

$$= \lim_{n\to\infty}\left(1 - \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\right)$$

$$= 1 - \frac{2+0+0}{6} = 1 - \frac{2}{6} = 1 - \frac{1}{3} = \frac{2}{3}$$