March 28 Math 1190 sec. 63 Spring 2017

Section 4.8: Antiderivatives; Differential Equations

Definition: A function F is called an antiderivative of f on an interval I if

$$F'(x) = f(x) \quad \text{for all } x \text{ in } I.$$

As a consequence of the Mean Value Theorem, we have ...

Theorem: If F is any antiderivative of f on an interval I, then the *most general* antiderivative of f on I is

$$F(x) + C \quad \text{where } C \text{ is an arbitrary constant.}$$

Find the most general antiderivative of f.

$$f(x) = \frac{4x^4+1}{x}, \quad I = (0, \infty)$$

We want to recognize f as the derivative of something.

We'll use algebra to make f recognizable.

$$f(x) = \frac{4x^4+1}{x} = \frac{4x^4}{x} + \frac{1}{x} = 4x^3 + \frac{1}{x}$$

$$F(x) = x^4 + \ln x + C$$

Check

$$F'(x) = 4x^3 + \frac{1}{x} + 0 = 4x^3 + \frac{1}{x} \quad \checkmark$$

Question: Find the most general antiderivative of f.

$$f(x) = \frac{3x-1}{x}, \quad I = (0, \infty)$$

(a) $F(x) = \dfrac{\frac{3x^2}{2} - x}{\frac{x^2}{2}} + C$ ← it is not legitimate to take the antiderivative of individual factors in a quotient or product.

(b) $F(x) = \dfrac{1}{x^2} + C$

(c) $F(x) = 3x - \ln x + C$

Find the most general antiderivative of

$$f(x) = x^n, \quad \text{where } n = 1, 2, 3, \ldots$$

We'll make a guess as to the form of an antiderivative,

$F(x) = Ax^k$ where A and k are constants.

We need $F'(x) = f(x) \implies Akx^{k-1} = x^n$

This gives $k-1 = n$ (match powers)

and $Ak = 1$ (match coefficients)

So $k = n+1$ and $A(n+1) = 1 \implies A = \frac{1}{n+1}$

This means that $F(x) = \frac{1}{n+1} x^{n+1}$

The most general anti derivative of x^n is

$$\boxed{\frac{x^{n+1}}{n+1} + C}$$

The power rule for anti derivatives.

This actually holds for all $n \neq -1$.

Some general results[1]:

(See the table on page 330 in Sullivan & Miranda for a more comprehensive list.)

Function	Particular Antiderivative	Function	Particular Antiderivative
$cf(x)$	$cF(x)$	$\cos x$	$\sin x$
$f(x)+g(x)$	$F(x)+G(x)$	$\sin x$	$-\cos x$
x^n, $n \neq -1$	$\frac{x^{n+1}}{n+1}$	$\sec^2 x$	$\tan x$
$\frac{1}{x}$	$\ln\|x\|$	$\csc x \cot x$	$-\csc x$
$\frac{1}{x^2+1}$	$\tan^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x$

[1] We'll use the term particular antiderivative to refer to any antiderivative that has no arbitrary constant in it.

Example

Find the most general antiderivative of $h(x) = x\sqrt{x}$ on $(0, \infty)$.

$h(x) = x\sqrt{x} = x \cdot x^{1/2} = x^{3/2}$

power rule

$\frac{x^{n+1}}{n+1}$

power rule w/ $n = \frac{3}{2}$, $\frac{3}{2} + 1 = \frac{5}{2}$

$H(x) = \frac{1}{5/2} x^{5/2} + C$

$$\boxed{H(x) = \frac{2}{5} x^{5/2} + C}$$

Example

Determine the function $H(x)$ that satisfies the following conditions

$$H'(x) = x\sqrt{x}, \quad \text{for all } x > 0, \text{ and } H(1) = 0.$$

this means that the point $(1,0)$ must be on H's graph

From the last example, H must have the form $H(x) = \frac{2}{5}x^{5/2} + C$.

We impose $H=0$ when $x=1$

$$0 = \frac{2}{5}(1)^{5/2} + C \implies 0 = \frac{2}{5} + C \implies C = \frac{-2}{5}$$

So our $\boxed{H(x) = \frac{2}{5}x^{5/2} - \frac{2}{5}}$.

Example

A **differential equation** is an equation that involves the derivative(s) of an unknown function. **Solving** such an equation would mean finding such an unknown function.

For example, we just solved the differential equation

$$\frac{dH}{dx} = x\sqrt{x}$$

subject to the additional condition that $H = 0$ when $x = 1$.

The condition $H(1) = 0$ is usually called an **initial condition** if x represents time. It may be called a **boundary condition** if x represents space.

Question

The **most general** solution to the differential equation

$$\frac{dy}{dx} = 2x + 1 \quad \text{is}$$

(a) $y = 2x + 1 + C$

(b) $y = x^2 + 1 + C$

(c) $y = x^2 + x$

(d) $y = x^2 + x + C$

Question

The solution to the differential equation subject to the boundary condition

$$\frac{dy}{dx} = 2x + 1 \qquad y(1) = -2$$

when $x=1$, $y=-2$

(a) $y = x^2 + x - 2$

(b) $y = x^2 + x - 4$

(c) $y = x^2 + x$

(d) $y = x^2 + x + C - 2$

$y = x^2 + x + C$

$-2 = 1^2 + 1 + C$

$-2 = 2 + C \Rightarrow C = -4$

Example

A particle moves along the x-axis so that its acceleration at time t is given by

$$a(t) = 12t - 2 \quad \text{m/sec}^2.$$

At time $t = 0$, the velocity v and position s of the particle are known to be

$$v(0) = 3 \quad \text{m/sec, and} \qquad s(0) = 4 \text{ m.}$$

Find the position $s(t)$ of the particle for all $t > 0$.

we know that $a(t) = \frac{dv}{dt}$

Power rule

$\frac{dv}{dt} = 12t - 2$ with $v(0) = 3$

$\frac{x^{n+1}}{n+1}$

$v = 12\frac{t^2}{2} - 2t + C = 6t^2 - 2t + C$

Impose $v(0)=3$ $\quad 3 = 6\cdot 0^2 - 2\cdot 0 + C \Rightarrow C = 3$

So the velocity $v(t) = 6t^2 - 2t + 3$ m/sec

Since $v(t) = \frac{ds}{dt}$, we have to solve

$$\frac{ds}{dt} = 6t^2 - 2t + 3 \quad \text{with} \quad s(0) = 4$$

$$s = 6\,\frac{t^3}{3} - 2\cdot\frac{t^2}{2} + 3t + C$$

$$s = 2t^3 - t^2 + 3t + C$$

Impose $s(0) = 4$

$$4 = 2\cdot 0^3 - 0^2 + 3\cdot 0 + C \implies C = 4$$

So the particle's position

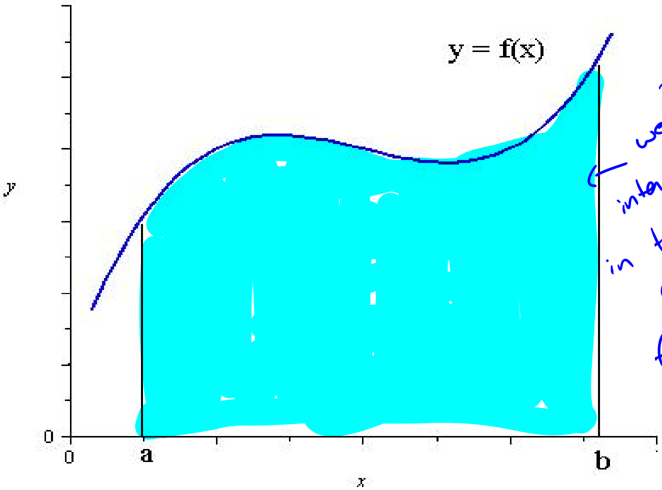
$$s(t) = 2t^3 - t^2 + 3t + 4 \quad m$$

Section 5.1: Area (under the graph of a nonnegative function)

We will investigate the area enclosed by the graph of a function f. We'll make the following assumptions (for now):

- f is continuous on the interval $[a, b]$, and
- f is nonnegative, i.e $f(x) \geq 0$, on $[a, b]$.

Our Goal: Find the area of such a region.



we're interested in the area of this flat region

Figure: Region under a positive curve $y = f(x)$ on an interval $[a, b]$.

We could approximate with rectangles determined by left end points.

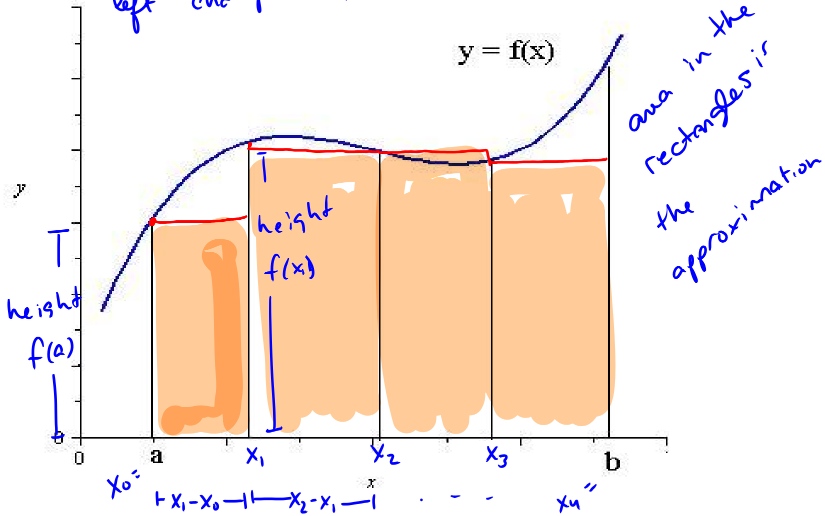


Figure: We could approximate the area by filling the space with rectangles.

We could use right end points instead.

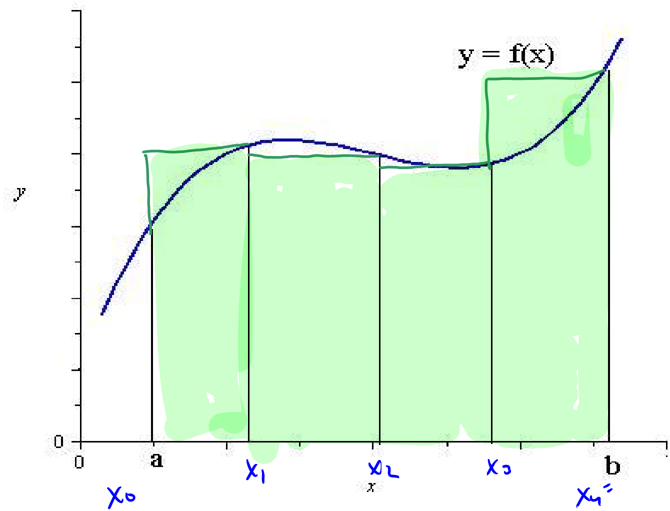


Figure: We could approximate the area by filling the space with rectangles.

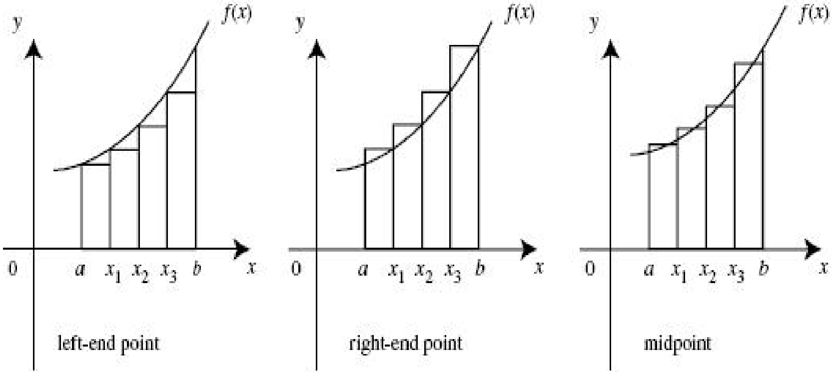


Figure: Some choices as to how to define the heights.

Approximating Area Using Rectangles

We can experiment with

- Which points to use for the heights (left, right, middle, other....)

- How many rectangles we use

to try to get a good approximation.

Definition: We will define the true area to be value we obtain taking the limit as the number of rectangles goes to $+\infty$.

Some terminology

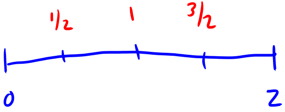
- A **Partition** P of an interval $[a, b]$ is a collection of points $\{x_0, x_1, ..., x_n\}$ such that
$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$
- A **Subinterval** is one of the intervals $x_{i-1} \leq x \leq x_i$ determined by a partition.
- The width of a subinterval is denoted $\Delta x_i = x_i - x_{i-1}$. If they are all the same size (equal spacing), then
$$\Delta x = \frac{b-a}{n}, \quad \text{and this is called the } \textbf{norm} \text{ of the partition.}$$
- A set of **sample points** is a set $\{c_1, c_2, \ldots, c_n\}$ such that $x_{i-1} \leq c_i \leq x_i$.

Taking the number of rectangles to ∞ is the same as taking the width $\Delta x \to 0$.

Example:

Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm Δx.

(a) For $n = 4$



$$\Delta x = \frac{b-a}{n} = \frac{2-0}{4} = \frac{1}{2}$$

$x_0 = 0$

$x_1 = \frac{1}{2}$

$x_2 = 1$

$x_3 = \frac{3}{2}$

$x_4 = 2$

Note

$x_0 = a$

$x_1 = a + 1 \cdot \Delta x$

$x_2 = a + 2\Delta x$

$x_3 = a + 3\Delta x$

In general

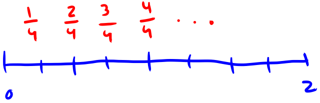
$x_i = a + i\Delta x$

$i = 0, 1, 2, \ldots, n$

Example:

Write an equally spaced partition of the interval $[0, 2]$ with the specified number of subintervals, and determine the norm Δx.

(b) For $n = 8$



$$\Delta x = \frac{b-a}{n} = \frac{2-0}{8} = \frac{1}{4}$$

$x_0 = 0$ $\quad x_4 = 1$ $\quad x_7 = 7/4$

$x_1 = \frac{1}{4}$ $\quad x_5 = \frac{5}{4}$ $\quad x_8 = 2$

$x_2 = \frac{1}{2}$ $\quad x_6 = \frac{3}{2}$

$x_3 = \frac{3}{4}$

Note

$$x_i = \underset{a}{0} + i\underset{\Delta x}{\left(\frac{1}{4}\right)}$$

Question

$n=6$

Write an equally spaced partition of the interval $[0, 2]$ with 6 subintervals, and determine the norm Δx.

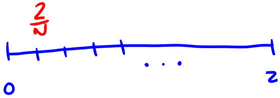
$\Delta x = \frac{2-0}{6} = \frac{1}{3}$

(a) $\left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right\}$ $\quad \Delta x = \frac{1}{3}$

(b) $\left\{0, \frac{1}{3}, \frac{2}{3}, 1, \frac{4}{3}, \frac{5}{3}, 2\right\}$ $\quad \Delta x = \frac{1}{6}$

(c) $\left\{0, \frac{1}{6}, \frac{1}{3}, 1, \frac{5}{6}, \frac{7}{6}, 2\right\}$ $\quad \Delta x = \frac{1}{3}$

(c) Find an equally spaced partition of $[0, 2]$ having N subintervals. What is the norm Δx?



$$\Delta x = \frac{b-a}{n} = \frac{2-0}{N} = \frac{2}{N}$$

$$x_0 = 0$$

$$x_1 = \frac{2}{N} = 0 + 1\cdot\Delta x$$

$$x_2 = 2\cdot\frac{2}{N} = 0 + 2\Delta x$$

$$x_3 = 3\cdot\frac{2}{N} = 0 + 3\Delta x$$

$$x_i = i\cdot\frac{2}{N} = \frac{2i}{N}$$

for $i = 0, 1, 2, \dots, N$

Note $x_N = N\cdot\frac{2}{N} = 2$ as required.

Approximating area with a Partition and sample points

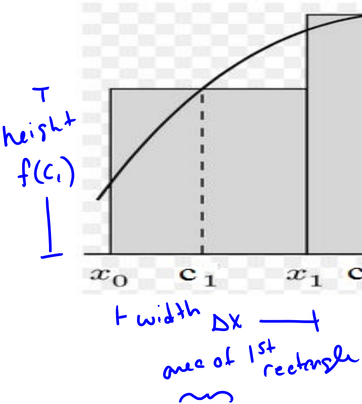


Figure: Area = $f(c_1)\Delta x + f(c_2)\Delta x + f(c_3)\Delta x + f(c_4)\Delta x$. This can be written as

$$\sum_{i=1}^{n} f(c_i)\Delta x. \quad \leftarrow \text{called a Riemann Sum}$$

Sum Notation

$\sum$ is the capital letter *sigma*, basically a capital Greek "S".

If $a_1, a_2, \ldots, a_n$ are a collection of real numbers, then

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n.$$

This is read as

the sum from i equals 1 *to n of* a_i *(a sub i).*

For example

$$\sum_{i=1}^{4} i = 1 + 2 + 3 + 4 = 10$$

$$\sum_{i=1}^{3} 2i^2 = 2 \cdot 1^2 + 2 \cdot 2^2 + 2 \cdot 3^2 = 28$$

In general, an equally spaced partition of $[a, b]$ with n subintervals means

- $\Delta x = \frac{b-a}{n}$
- $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, i.e. $x_i = a + i\Delta x$
- Taking heights to be

height width

$$\text{left ends} \quad c_i = x_{i-1} \quad \text{area} \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$\text{right ends} \quad c_i = x_i \quad \text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

- The true area exists (for f continuous) and is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

Lower and Upper Sums

The standard way to set up these sums is to take c_i such that

$$f(c_i) \quad \text{is the abs. minimum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set A_L

$$A_L = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

This is called a **Lower Riemann sum**.

Lower and Upper Sums

Then, we take C_i such that

$$f(C_i) \quad \text{is the abs. maximum value of } f \text{ on } [x_{i-1}, x_i]$$

Then set A_U

$$A_U = \lim_{n\to\infty} \sum_{i=1}^{n} f(C_i)\Delta x.$$

This is called a **Upper Riemann sum**.

Lower and Upper Sums

If f is continuous on $[a, b]$, then it will necessarily be that

$$A_L = A_U.$$

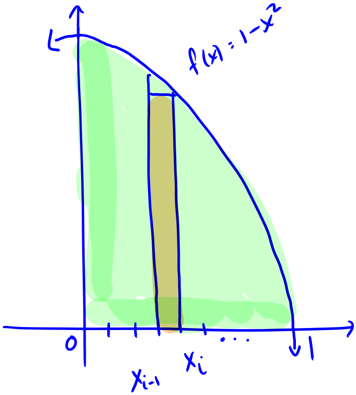
This value is the true area.

In practice, these are tough to compute unless f is only increasing or only decreasing. So instead, we tend to use left and right sums.

Example: Find the area under the curve $f(x) = 1 - x^2$, $0 \le x \le 1$.

Use right end points $c_i = x_i$ and assume the following identity

$$\sum_{i=1}^{n} i^2 = \frac{2n^3 + 3n^2 + n}{6} \quad \text{(sum of first } n \text{ squares)}$$



Form a partition w/ n subintervals

$$\Delta x = \frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n}$$

$$x_0 = 0, \quad x_1 = \frac{1}{n}, \quad x_2 = 2 \cdot \frac{1}{n}, \ldots$$

$$x_i = a + i\Delta x = 0 + i\frac{1}{n}$$

$$x_i = \frac{i}{n}$$

for one rectangle, the height is $f(x_i)$

$$f(x_i) = 1 - (x_i)^2 = 1 - \left(\frac{i}{n}\right)^2 = 1 - \frac{i^2}{n^2}$$

The width is $\Delta x = \frac{1}{n}$

One rectangle area height · width

$f(x_i)$ Δx

$$\left(1 - \frac{i^2}{n^2}\right) \cdot \frac{1}{n}$$

$$\text{area} \approx \sum_{i=1}^{n} f(x_i)\,\Delta x = \sum_{i=1}^{n} \left(1 - \frac{i^2}{n^2}\right) \cdot \frac{1}{n}$$

$$= \sum_{i=1}^{n} \left(\frac{1}{n} - \frac{i^2}{n^3}\right)$$

$$= \sum_{i=1}^{n} \frac{1}{n} - \sum_{i=1}^{n} \frac{i^2}{n^3}$$

Note $$\sum_{i=1}^{n} \frac{1}{n} = \underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}} = n\left(\frac{1}{n}\right) = 1$$

Note $\sum_{i=1}^{n} \frac{1}{n} = \underbrace{\frac{1}{n} + \frac{1}{n} + \frac{1}{n} + \cdots + \frac{1}{n}}_{n \text{ times}} = n\left(\frac{1}{n}\right) = 1$

and $\sum_{i=1}^{n} \frac{i^2}{n^3} = \frac{1}{n^3}\left(1^2 + 2^2 + \cdots + n^2\right) = \frac{1}{n^3} \frac{2n^3 + 3n^2 + n}{6}$

So we have

$$\text{area} \approx 1 - \frac{2n^3 + 3n^2 + n}{6n^3}$$

To get the true area, take $n \to \infty$

$$\text{area} = \lim_{n\to\infty}\left(1 - \frac{2n^3+3n^2+n}{6n^3}\right)$$

$$= \lim_{n\to\infty}\left(1 - \frac{2n^3+3n^2+n}{6n^3}\left(\frac{\frac{1}{n^3}}{\frac{1}{n^3}}\right)\right)$$

$$= \lim_{n\to\infty}\left(1 - \frac{2+\frac{3}{n}+\frac{1}{n^2}}{6}\right)$$

$$= 1 - \frac{2+0+0}{6} = 1 - \frac{2}{6} = 1 - \frac{1}{3} = \frac{2}{3}$$