March 29 Math 2306 sec 59 Spring 2016

Section 11: Linear Mechanical Equations

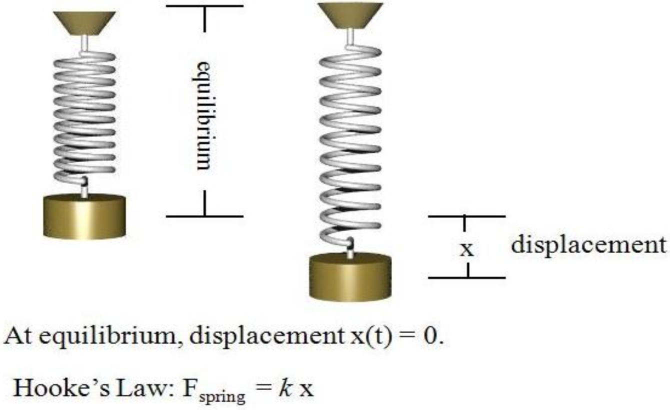


Figure: In the absence of any displacement, the system is at equilibrium. Displacement $x(t)$ is measured from equilibrium $x = 0$.

Building an Equation: Hooke's Law

Newton's Second Law: $F = ma$ Force = mass times acceleration

$$a = \frac{d^2x}{dt^2} \quad \Longrightarrow \quad F = m\frac{d^2x}{dt^2}$$

Hooke's Law: $F = kx$ Force exerted by the spring is proportional to displacement

The force imparted by the spring opposes the direction of motion.

$$m\frac{d^2x}{dt^2} = -kx \quad \Longrightarrow \quad x'' + \omega^2 x = 0 \quad \text{where} \quad \omega = \sqrt{\frac{k}{m}}$$

Convention We'll Use: Up will be positive ($x > 0$), and down will be negative ($x < 0$). This orientation is arbitrary and follows the convention in Trench.

Appropriate Units

The two unit systems, US customary and SI, require quantities in certain units.

force/ weight		pounds (lb)
length		feet (ft)
time		seconds (sec)
mass		slugs

Table: US units

force/ weight		Newtons (N)
length		meters (m)
time		seconds (sec)
mass		kilograms (kg)

Table: SI units

Deducing Constants from Givens

We may need to analyze a problem and deduce various constants:

- If an object of weight W stretches a spring δx units, the spring constant k satisfies

$$W = k\delta x \quad \Longrightarrow \quad k = \frac{W}{\delta x} \qquad \text{N/m, or lb/ft}$$

- Given a mass (kg or slugs), we can deduce weight (N or lb)—or vice versa

$$m = \frac{W}{g} \quad \Longleftrightarrow \quad W = mg \qquad g\text{, accel. due to gravity}$$

Obtaining ω: *Displacment in Equilibrium*

Let $\delta x =$ length of spring w/ object $-$ length of spring w/o object.

Displacement in equilibrium: Applying Hooke's law with the weight as force, we have

$$mg = k\delta x.$$

We observe that the value ω can be deduced from δx by

$$\omega^2 = \frac{k}{m} = \frac{g}{\delta x}.$$

This allows us to compute ω^2 without knowing m or k IF δx is known.

Simple Harmonic Motion

$$x'' + \omega^2 x = 0, \quad x(0) = x_0, \quad x'(0) = x_1 \tag{1}$$

Here, x_0 and x_1 are the initial position (relative to equilibrium) and velocity, respectively. The solution is

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

called the **equation of motion.**

Note that equation (1) is a second order constant coefficient equation. We can solve this IVP. Hence there is NO NEED to memorize the solution!

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t)$$

Characteristics of the system include:

- the period $T = \frac{2\pi}{\omega}$,
- the frequency $f = \frac{1}{T} = \frac{\omega}{2\pi}$[1]
- the circular (or angular) frequency ω, and
- the amplitude or maximum displacement $A = \sqrt{x_0^2 + (x_1/\omega)^2}$

[1] Various authors call f the natural frequency and others use this term for ω.

Amplitude and Phase Shift

We can formulate the solution in terms of a single sine (or cosine) function. Letting

$$x(t) = x_0 \cos(\omega t) + \frac{x_1}{\omega} \sin(\omega t) = A \sin(\omega t + \phi)$$

requires

$$A = \sqrt{x_0^2 + (x_1/\omega)^2},$$

and the **phase shift** ϕ must be defined by

$$\sin\phi = \frac{x_0}{A}, \quad \text{with} \quad \cos\phi = \frac{x_1}{\omega A}.$$

Initial Conditions

To have an IVP, initial information about position and velocity must be given. We'll keep in mind that

- Up is positive, and down is negative—this is the convention we'll use.
- If an object starts **at equilibrium**, then initial postion $x(0) = 0$.
- If an object starts **from rest**, then initial velocity $x'(0) = 0$.

Example

An object stretches a spring 6 inches in equilibrium. Assuming no driving force and no damping, set up the differential equation describing this system.

Our ODE should look like $x'' + \omega^2 x = 0$

where $\omega^2 = \frac{k}{m}$ (spring const. / mass)

we don't know k or m, but the displacement "in equilibrium" is given.

$\delta x = 6 \text{ in} = \frac{1}{2} \text{ ft}$

For US units, $g = 32$ ft/sec^2

$$\omega^2 = \frac{g}{\delta x} = \frac{32 \text{ ft/sec}^2}{\frac{1}{2} \text{ ft}} = 64 \frac{1}{\text{sec}^2}$$

Our ODE is

$$x'' + 64x = 0$$

Example

A 4 pound weight stretches a spring 6 inches. The mass is released from a position 4 feet above equilibrium with an initial downward velocity of 24 ft/sec. Find the equation of motion, the period, amplitude, phase shift, and frequency of the motion. (Take $g = 32$ ft/sec^2.)

Our ODE will look like $mx'' + kx = 0$ i.e. $x'' + \omega^2 x = 0$

We need m and k.

We know that $W = mg$, $W = 4\,\text{lb}$ so

$$4\,\text{lb} = m\left(32\ \text{ft}/\text{sec}^2\right) \Rightarrow m = \frac{4\,\text{lb}}{32\ \text{ft}/\text{sec}^2} = \frac{1}{8}\ \text{Slugs}$$

4lb force stretches the spring 6 in. By Hooke's Law

$$W = k\delta x \implies k = \frac{W}{\delta x} = \frac{4\,\text{lb}}{\frac{1}{2}\,\text{ft}} = 8\ \frac{\text{lb}}{\text{ft}}$$

So $mx'' + kx = 0 \implies \frac{1}{8}x'' + 8x = 0$

In standard form $\quad x'' + 64x = 0$

From the statement, $x(0) = 4$ and $x'(0) = -24$

above equilibrium hence positive.

downward hence negative

2^{nd} order, linear, constant coeff, homogeneous eqn.

The characteristic eqn is

$$r^2+64=0 \Rightarrow r^2=-64 \Rightarrow r=\pm 8i$$
$$=0\pm 8i$$

$\alpha=0,\ \beta=8$

$x_1=e^{0t}\cos(8t)=\cos(8t)$

$x_2=e^{0t}\sin(8t)=\sin(8t)$

$$\boxed{x(t)=c_1\cos(8t)+c_2\sin(8t)}$$

$$x'(t)=-8c_1\sin(8t)+8c_2\cos(8t)$$

$$x(0)=4 \Rightarrow x(0)=c_1\cos 0+c_2\sin 0=4 \Rightarrow c_1=4$$

$$x'(0)=-24 \Rightarrow x'(0)=-8c_1\sin 0+8c_2\cos 0=-24 \Rightarrow c_2=\frac{-24}{8}=-3$$

The position is

$$\boxed{x(t)=4\cos(8t)-3\sin(8t).}$$

Properties: Period $T=\frac{2\pi}{\omega}=\frac{2\pi}{8}=\frac{\pi}{4}$

frequency $f=\frac{1}{T}=\frac{4}{\pi}$

Amplitude $A=\sqrt{c_1^2+c_2^2}=\sqrt{4^2+(-3)^2}=5$

The phase shift ϕ satisfies

$$\sin\phi = \frac{x_0}{A} = \frac{4}{5} \quad \text{and} \quad \cos\phi = \frac{x_1/\omega}{A} = \frac{-3}{5}$$

Note $\sin\phi > 0$ and $\cos\phi < 0$

so ϕ is a quad II angle.

We can use the inverse cosine function to find ϕ since its range is $[0, \pi]$.

$$\phi = \cos^{-1}\left(\frac{-3}{5}\right) \approx 2.21$$ roughly $127°$

Free Damped Motion

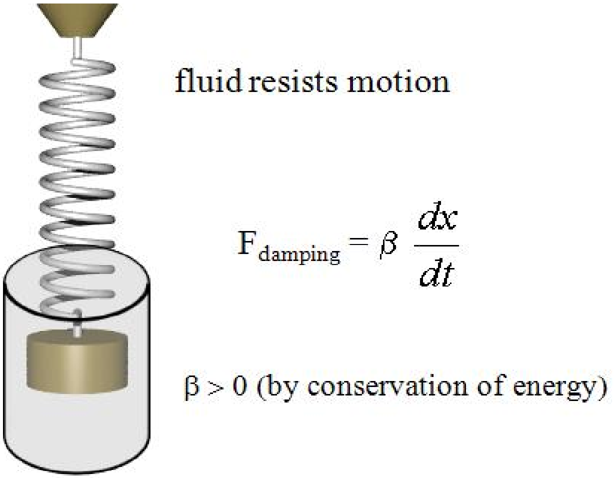


Figure: If a damping force is added, we'll assume that this force is proportional to the instantaneous velocity.

Free Damped Motion

Now we wish to consider an added force corresponding to damping—friction, a dashpot, air resistance.

Total Force = Force of spring + Force of damping

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx \quad \Longrightarrow \quad \frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0$$

where

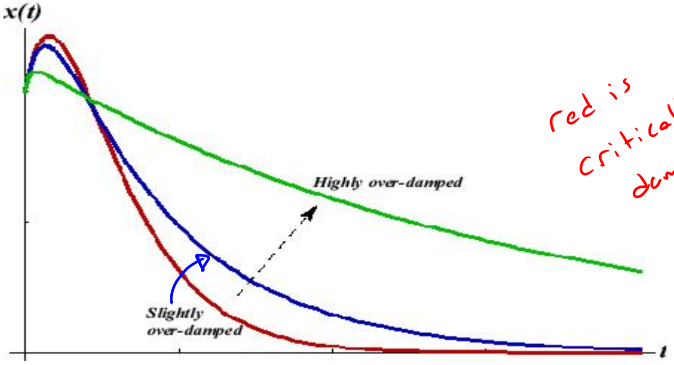
$$2\lambda = \frac{\beta}{m} \quad \text{and} \quad \omega = \sqrt{\frac{k}{m}}.$$

Three qualitatively different solutions can occur depending on the nature of the roots of the characteristic equation

$$r^2 + 2\lambda r + \omega^2 = 0 \quad \text{with roots} \quad r_{1,2} = -\lambda \pm \sqrt{\lambda^2 - \omega^2}.$$

Case 1: $\lambda^2 > \omega^2$ Overdamped

$$x(t) = e^{-\lambda t}\left(c_1 e^{t\sqrt{\lambda^2-\omega^2}} + c_2 e^{-t\sqrt{\lambda^2-\omega^2}}\right)$$



red is critically damped.

Figure: Two distinct real roots. No oscillations. Approach to equilibrium may be slow.

Case 2: $\lambda^2 = \omega^2$ Critically Damped

$$x(t) = e^{-\lambda t}(c_1 + c_2 t)$$

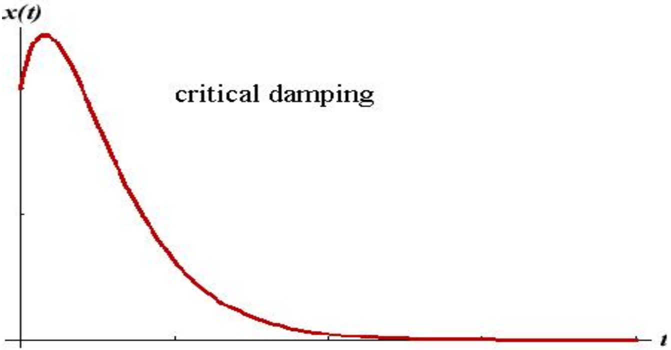


Figure: One real root. No oscillations. Fastest approach to equilibrium.

Case 3: $\lambda^2 < \omega^2$ Underdamped

$$x(t) = e^{-\lambda t}\left(c_1 \cos(\omega_1 t) + c_2 \sin(\omega_1 t)\right), \quad \omega_1 = \sqrt{\omega^2 - \lambda^2}$$

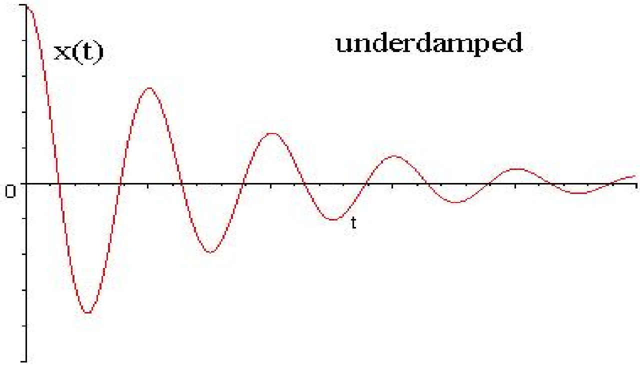
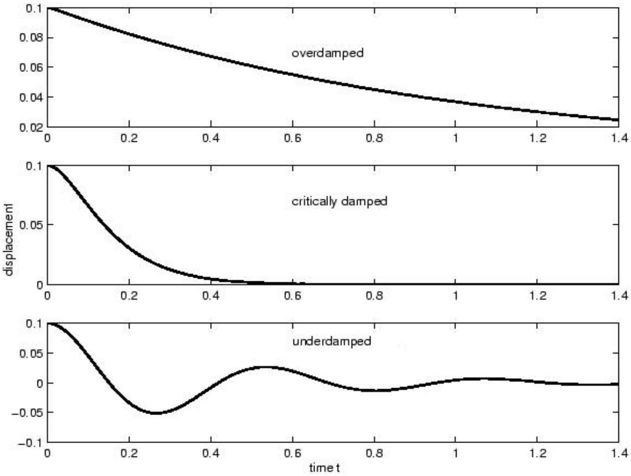


Figure: Complex conjugate roots. Oscillations occur as the system approaches (resting) equilibrium.

Comparison of Damping



- 2 real roots

- one repeated root

- complex roots

Figure: Comparison of motion for the three damping types.

Example

A 2 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 10 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped.

Our ODE will look like $mx'' + \beta x' + kx = 0$

this is β times the instantaneous velocity

From the statement

$m = 2 \qquad \beta = 10 \qquad k = 12$

The ODE is $2x'' + 10x' + 12x = 0$, in standard form

$$\boxed{x'' + 5x' + 6x = 0}$$

The characteristic eqn is $r^2 + 5r + 6 = 0$

$$(r+2)(r+3) = 0$$

$$\Rightarrow r = -2 \text{ or } r = -3$$

2 real distinct roots means the system is over damped.

For grins and giggles, note that

$$\omega^2 = 6 \quad \text{and} \quad 2\lambda = 5 \quad \text{so} \quad \lambda = 5/2$$

$$\lambda^2 - \omega^2 = (5/2)^2 - 6 = \frac{25}{4} - \frac{24}{4} = \frac{1}{4} > 0$$

We would again conclude that it is

over damped.

Example

A 3 kg mass is attached to a spring whose spring constant is 12 N/m. The surrounding medium offers a damping force numerically equal to 12 times the instantaneous velocity. Write the differential equation describing this system. Determine if the motion is underdamped, overdamped or critically damped. If the mass is released from the equilibrium position with an upward velocity of 1 m/sec, solve the resulting initial value problem.

We know that $mx'' + \beta x' + kx = 0$

Here $m = 3, \quad \beta = 12, \quad k = 12$

Standard form

$$3x'' + 12x' + 12x = 0 \implies x'' + 4x' + 4x = 0$$

The ch. eqn is $r^2+4r+4=0$

$(r+2)^2=0 \Rightarrow r=-2$ repeated

$$\boxed{\text{One repeated root} \Rightarrow \text{Critically damped}}$$

We want to solve the IVP

from equilibrium

upward so positive

$x''+4x'+4x=0,\quad x(0)=0 \quad x'(0)=1$

From $r=-2$, $x_1(t)=e^{-2t}$ and $x_2(t)=te^{-2t}$

So $\quad x(t) = c_1 e^{-2t} + c_2 t e^{-2t}$

$$x'(t) = -2c_1 e^{-2t} + c_2 e^{-2t} - 2c_2 t e^{-2t}$$

$x(0) = 0 \quad \Rightarrow \quad c_1 e^{0} + c_2 \cdot 0 e^{0} = 0 \quad \Rightarrow \quad c_1 = 0$

$x'(0) = 1 \quad \Rightarrow \quad c_2 e^{0} - 2c_2 \cdot 0 e^{0} = 1 \quad \Rightarrow \quad c_2 = 1$

The solution to the IVP is

$$\boxed{x(t) = t e^{-2t}}$$

Driven Motion

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \geq 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and } (2) \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

$x'' + \omega^2 x = F_0 \sin(\gamma t)$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ If $\gamma \neq \omega$

No duplication of x_c. This guess works.

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

Note that

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

Using the method of undetermined coefficients, the **first guess** to the particular solution is

$x_p = A\cos(\gamma t) + B\sin(\gamma t)$ If $\gamma = \omega$

we need

$$x_p = At\cos(\gamma t) + Bt\sin(\gamma t)$$