March 29 Math 3260 sec. 55 Spring 2018

Section 6.1: Inner Product, Length, and Orthogonality

We defined the inner product on $\mathbb{R}^n$:

Definition: For vectors **u** and **v** in $\mathbb{R}^n$ we define the **inner product** of **u** and **v** (also called the **dot product**) by the **matrix product**

$$\mathbf{u}^T\mathbf{v} = [u_1\ u_2\ \cdots\ u_n]\begin{bmatrix} v_1\\ v_2\\ \vdots\\ v_n\end{bmatrix} = u_1v_1 + u_2v_2 + \cdots + u_nv_n.$$

We'll use the notation $\quad \mathbf{u}\cdot\mathbf{v} \ = \ \mathbf{u}^T\mathbf{v}.$

Properties of the Inner Product & Norm

Theorem: For $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ in $\mathbb{R}^n$ and real scalar c

(a) $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$

(b) $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w}$

(c) $c(\mathbf{u} \cdot \mathbf{v}) = (c\mathbf{u}) \cdot \mathbf{v} = \mathbf{u} \cdot (c\mathbf{v})$

(d) $\mathbf{u} \cdot \mathbf{u} \geq 0$, with $\mathbf{u} \cdot \mathbf{u} = 0$ if and only if $\mathbf{u} = \mathbf{0}$.

Definition: The **norm** of the vector $\mathbf{v}$ in $\mathbb{R}^n$ is the nonnegative number denoted and defined by

$$\|\mathbf{v}\| = \sqrt{\mathbf{v} \cdot \mathbf{v}} = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$

where $v_1, v_2, \ldots, v_n$ are the components of $\mathbf{v}$.

Unit Vectors and Normalizing

Theorem: For vector $\mathbf{v}$ in $\mathbb{R}^n$ and scalar c

$$\|c\mathbf{v}\| = |c|\|\mathbf{v}\|.$$

Definition: A vector $\mathbf{u}$ in $\mathbb{R}^n$ for which $\|\mathbf{u}\| = 1$ is called a **unit vector.**

Remark: Given any nonzero vector $\mathbf{v}$ in $\mathbb{R}^n$, we can obtain a unit vector $\mathbf{u}$ in the same direction as $\mathbf{v}$

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

This process, of dividing out the norm, is called **normalizing** the vector $\mathbf{v}$.

Distance in $\mathbb{R}^n$ & Orthogonality

Definition: For vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$, the **distance between u and v** is denoted and defined by

$$\text{dist}(\mathbf{u}, \mathbf{v}) = \|\mathbf{u} - \mathbf{v}\|.$$

Definition: Two vectors are $\mathbf{u}$ and $\mathbf{v}$ **orthogonal** if $\mathbf{u} \cdot \mathbf{v} = 0$.

Perpendicular: If nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ are perpendicular in $\mathbb{R}^n$, then $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{u} + \mathbf{v}\|$. This is the case if and only if $\mathbf{u} \cdot \mathbf{v} = 0$.

The Pythagorean Theorem: Two vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal if and only if

$$\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2.$$

Orthogonal Complement

Definition: Let W be a subspace of $\mathbb{R}^n$. A vector $\mathbf{z}$ in $\mathbb{R}^n$ is said to be **orthogonal to** W if $\mathbf{z}$ is orthogonal to every vector in W.

$\vec{z} \cdot \vec{w} = 0$ for every $\vec{w}$ in W

Definition: Given a subspace W of $\mathbb{R}^n$, the set of all vectors orthogonal to W is called the **orthogonal complement** of W and is denoted by

$$W^{\perp}.$$

Read as "W perp"

Theorem:

$W^\perp$ is a subspace of $\mathbb{R}^n$.

We need to show that $\vec{0}$ is in $W^\perp$ and that $W^\perp$ is closed under vector addition and scalar multiplication.

Since $\vec{0}\cdot\vec{w}=0$ for every $\vec{w}$ in W, $\vec{0}$ is in $W^\perp$.

To see that it's closed under vector addition, suppose $\vec{u}$ and $\vec{v}$ are in $W^\perp$. Then

$$\vec{u}\cdot\vec{w}=0 \quad \text{and} \quad \vec{v}\cdot\vec{w}=0 \quad \text{for every } \vec{w} \text{ in } W.$$

Note that for any $\vec{w}$ in W

$$(\vec{u}+\vec{v})\cdot\vec{w} = \vec{u}\cdot\vec{w}+\vec{v}\cdot\vec{w} = 0+0 = 0.$$

So $\vec{u}+\vec{v}$ is in $W^\perp$ which is closed under vector addition.

Letting c be any scalar and $\vec{w}$ any vector in W,

$$(c\vec{u})\cdot\vec{w} = c(\vec{u}\cdot\vec{w}) = c(0) = 0.$$

Hence $c\vec{u}$ is in $W^\perp$ which is closed under scalar multiplication.

It follows that $W^\perp$ is a subspace of $\mathbb{R}^n$.

Example

Let W =Span$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$. Show that $W^\perp$ =Span$\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

Give a geometric interpretation of W and $W^\perp$ as subspaces of $\mathbb{R}^3$.

If $\vec{w}$ is any vector in W, then

$$\vec{w} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix}$$ for some scalars a, b.

If $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$ is orthogonal to W, then $\vec{u} \cdot \vec{w} = 0$.

$$\vec{u} \cdot \vec{w} = u_1 a + u_2(0) + u_3 b = 0$$ for <u>any</u> a, b.

$$u_1 a + u_3 b = 0 \qquad \text{for all choices of } a, b.$$

If $a=1$ and $b=0$, we conclude that

$$u_1 \cdot 1 + u_3 \cdot 0 = 0 \implies u_1 = 0.$$

If $a=0$ and $b=1$, we conclude that

$$u_1 \cdot 0 + u_3 \cdot 1 = 0 \implies u_3 = 0.$$

So $\vec{u}$ has to have the form

$$\vec{u} = \begin{bmatrix} 0 \\ u_2 \\ 0 \end{bmatrix} = u_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$ with u_2 any real number.

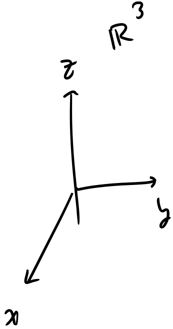
So $$W^{\perp} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

As for geometry, $$W = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This is the xz-plane; all points $(x, 0, z)$.

And $$W^{\perp} = \text{Span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

This is the y-axis; all points $(0, y, 0)$.



Example

Let $A = \begin{bmatrix} 1 & 3 & 2 \\ -2 & 0 & 4 \end{bmatrix}$. Show that if $\mathbf{x}$ is in Nul(A), then $\mathbf{x}$ is in $[\text{Row}(A)]^{\perp}$.

$$A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & \frac{4}{3} \end{bmatrix}$$

For $\vec{x}$ in Nul A

$x_1 = 2x_3$

$x_2 = \frac{-4}{3} x_3$

x_3 - free

$\vec{x}$ in Nul A $\quad \vec{x} = x_3 \begin{bmatrix} 2 \\ \frac{-4}{3} \\ 1 \end{bmatrix}$

Also $\quad$ Row A = Span $\left\{ \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ \frac{4}{3} \end{bmatrix} \right\}$.

Letting, $\vec{w}$ be in Row A. Then

$$\vec{w} = a\begin{bmatrix}1\\0\\-2\end{bmatrix} + b\begin{bmatrix}0\\1\\\frac{4}{3}\end{bmatrix}$$ for some scalers a, b

For $\vec{x}$ in Nul A, $\vec{x} = c\begin{bmatrix}2\\-\frac{4}{3}\\1\end{bmatrix}$.

$$\vec{x}\cdot\vec{w} = c\begin{bmatrix}2\\-\frac{4}{3}\\1\end{bmatrix}\cdot\left(a\begin{bmatrix}1\\0\\-2\end{bmatrix} + b\begin{bmatrix}0\\1\\4/3\end{bmatrix}\right)$$

$$= ac \begin{bmatrix} 2 \\ -\frac{4}{3} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix} + cb \begin{bmatrix} 2 \\ -\frac{4}{3} \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 4/3 \end{bmatrix}$$

$$= ac\,(2-2) + cb\left(-\frac{4}{3} + \frac{4}{3}\right)$$

$$= 0 + 0$$

$$= 0$$

Everything in $\mathrm{Nul}\,A$ is orthogonal to everything in $\mathrm{Row}\,A$. Hence $\mathrm{Nul}\,A = [\mathrm{Row}\,A]^{\perp}$

Theorem

Theorem: Let A be an $m \times n$ matrix. The orthogonal complement of the row space of A is the null space of A. That is

$$[\text{Row}(A)]^\perp \;=\; \text{Nul}(A).$$

The orthongal complement of the column space of A is the null space of A^T—i.e.

$$[\text{Col}(A)]^\perp \;=\; \text{Nul}(A^T).$$

Example: Find the orthogonal complement of Col(A)

$$A = \begin{bmatrix} 5 & 2 & 1 \\ -3 & 3 & 0 \\ 2 & 4 & 1 \\ 2 & -2 & 9 \\ 0 & 1 & -1 \end{bmatrix}$$

We can use that

$$[\text{Col} A]^{\perp} = \text{Nul}(A^T).$$

$$A^T \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 & -54 & 7 \\ 0 & 1 & 0 & -\frac{146}{3} & 19/3 \\ 0 & 0 & 1 & 63 & -8 \end{bmatrix}$$

For $A^T\vec{x} = \vec{0}$

$$x_1 = 54x_4 - 7x_5$$

$$x_2 = \frac{146}{3}x_4 - \frac{19}{3}x_5 \qquad x_4, x_5 \text{ free}$$

$$x_3 = -63x_4 + 8x_5$$

$$\vec{x} = x_4 \begin{bmatrix} 54 \\ \frac{146}{3} \\ -63 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -7 \\ -\frac{19}{3} \\ 8 \\ 0 \\ 1 \end{bmatrix}$$

So

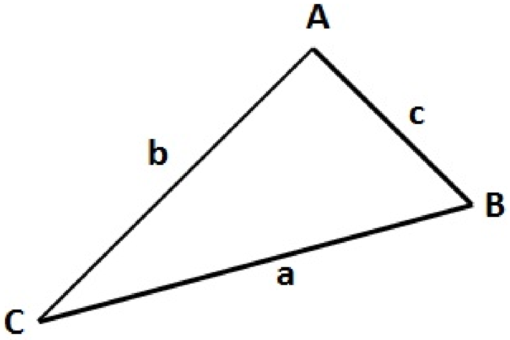
$$\left[\operatorname{Col} A\right]^{\perp} = \operatorname{Span}\left\{ \begin{bmatrix} 54 \\ \frac{146}{3} \\ -63 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -7 \\ -\frac{19}{3} \\ 8 \\ 0 \\ 1 \end{bmatrix} \right\}$$

This is actually a basis.

Recall the Law of Cosines

For triangle with angles A, B, C and opposite sides of lengths a, b, and c, respectively,

$$c^2 = a^2 + b^2 - 2ab\cos(C)$$



Geometry of the Dot Product

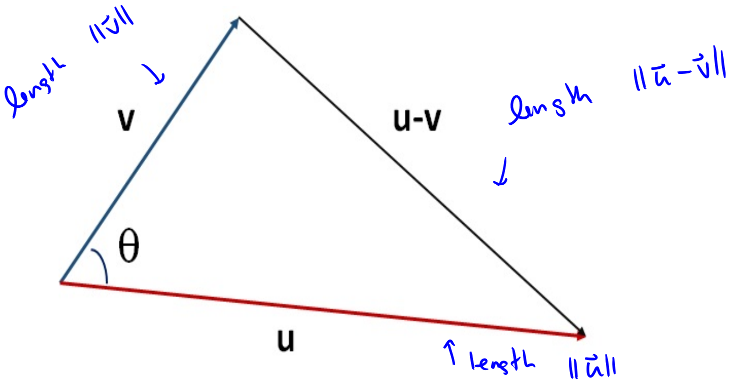


Figure: We can use the law of cosines to show that in $\mathbb{R}^2$ that $\mathbf{u}\cdot\mathbf{v}$ is related to the angle between the two (nonzero) vectors. This holds in $\mathbb{R}^n$. We're just restricting n to 2 for ease of computation.

By the law of Cosines

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\|\vec{u}\|\,\|\vec{v}\| \cos\theta$$

From before, we know that

$$\|\vec{u}-\vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2 - 2\vec{u}\cdot\vec{v}.$$

Comparing

$$\vec{u}\cdot\vec{v} = \|\vec{u}\|\,\|\vec{v}\| \cos\theta$$

If $\vec{u}$ and $\vec{v}$ are nonzero, the angle between them satisfies

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}.$$

Section 6.2: Orthogonal Sets

Remark: We know that if $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ is a basis for a subspace W of $\mathbb{R}^n$, then each vector $\mathbf{x}$ in W can be realized (uniquely) as a sum

$$\mathbf{x} = c_1\mathbf{b}_2 + \cdots + c_p\mathbf{b}_p.$$

If n is very large, the computations needed to determine the coefficients $c_1, \ldots, c_p$ may require a lot of time (and machine memory).

Question: Can we seek a basis whose nature simplifies this task? And what properties should such a basis possess?

Orthogonal Sets

Definition: An indexed set $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in $\mathbb{R}^n$ is said to be an **orthogonal set** provided each pair of distinct vectors in the set is orthogonal. That is, provided

$$\mathbf{u}_i \cdot \mathbf{u}_j = 0 \quad \text{whenever} \quad i \neq j.$$

Example: Show that the set $\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$ is an orthogonal set.

Label them $\vec{u}_1$ $\vec{u}_2$ $\vec{u}_3$

$$\vec{u}_1 \cdot \vec{u}_2 = 3(-1) + 1(2) + 1(1) = -3 + 3 = 0$$

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$$

$$\vec{u}_1 \cdot \vec{u}_3 = 3(-1) + 1(-4) + 1(7) = -7 + 7 = 0$$

$$\vec{u}_2 \cdot \vec{u}_3 = (-1)(-1) + 2(-4) + 1(7) = 8 - 8 = 0$$

Orthogonal Basis

Definition: An **orthogonal basis** for a subspace W of $\mathbb{R}^n$ is a basis that is also an orthogonal set.

Theorem: Let $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$. Then each vector $\mathbf{y}$ in W can be written as the linear combination

$$\mathbf{y} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + \cdots + c_p\mathbf{u}_p, \quad \text{where the weights}$$

$$c_j = \frac{\mathbf{y} \cdot \mathbf{u}_j}{\mathbf{u}_j \cdot \mathbf{u}_j}. \quad = \frac{\vec{y} \cdot \vec{u}_j}{\|\vec{u}_j\|^2}$$

$\vec{u}_1 \qquad \vec{u}_2 \qquad \vec{u}_3$

Example

$$\left\{ \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -4 \\ 7 \end{bmatrix} \right\}$$ is an orthogonal basis of $\mathbb{R}^3$. Express the vector $\mathbf{y} = \begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix}$ as a linear combination of the basis vectors.

We need $\|u_j\|^2$ and $\vec{y} \cdot \vec{u}_j$

$\|\vec{u}_1\|^2 = 3^2 + 1^2 + 1^2 = 11$

$\|\vec{u}_2\|^2 = (-1)^2 + 2^2 + 1^2 = 6$

$\|\vec{u}_3\|^2 = (-1)^2 + (-4)^2 + 7^2 = 66$

$\vec{y} \cdot \vec{u}_1 = -6 + 3 = -3$

$\vec{y} \cdot \vec{u}_2 = 2 + 6 = 8$

$\vec{y} \cdot \vec{u}_3 = 2 - 12 = -10$

So

$$\vec{y} = \frac{-3}{11}\vec{u}_1 + \frac{8}{6}\vec{u}_2 - \frac{10}{66}\vec{u}_3$$

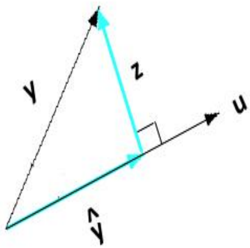
$$\vec{y} = \frac{-3}{11}\vec{u}_1 + \frac{4}{3}\vec{u}_2 - \frac{5}{33}\vec{u}_3$$

Projection

Given a nonzero vector **u**, suppose we wish to decompose another nonzero vector **y** into a sum of the form

$$\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$$

in such a way that $\hat{\mathbf{y}}$ is parallel to **u** and **z** is perpendicular to **u**.



$\hat{y}$ is the "part of $\vec{y}$ in the direction of $\vec{u}$"

Projection

Since $\hat{\mathbf{y}}$ is parallel to $\mathbf{u}$, there is a scalar α such that

$$\hat{\mathbf{y}} = \alpha\mathbf{u}.$$

We have $\vec{y} = \hat{y} + \vec{z}$ with $\hat{y} = \alpha\vec{u}$ and $\vec{z}$ perpendicular to $\vec{u}$.

$$\vec{y} = \alpha\vec{u} + \vec{z}$$

$$\vec{u}\cdot\vec{y} = \vec{u}\cdot(\alpha\vec{u} + \vec{z}) = \alpha\,\vec{u}\cdot\vec{u} + \underbrace{\vec{u}\cdot\vec{z}}_{=0}$$

$$\Rightarrow \quad \vec{u}\cdot\vec{y} = \alpha(\vec{u}\cdot\vec{u})$$

$$\Rightarrow \quad \alpha = \frac{\vec{u}\cdot\vec{y}}{\vec{u}\cdot\vec{u}}$$

Projection onto the subspace $L = \mathrm{Span}\{\mathbf{u}\}$

$$\textbf{Notation:} \quad \hat{\mathbf{y}} = \mathrm{proj}_L = \left(\frac{\mathbf{y}\cdot\mathbf{u}}{\mathbf{u}\cdot\mathbf{u}}\right)\mathbf{u}$$

Example: Let $\mathbf{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$ and $\mathbf{u} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Write $\mathbf{y} = \hat{\mathbf{y}} + \mathbf{z}$ where $\hat{\mathbf{y}}$ is in Span$\{\mathbf{u}\}$ and $\mathbf{z}$ is orthogonal to $\mathbf{u}$.

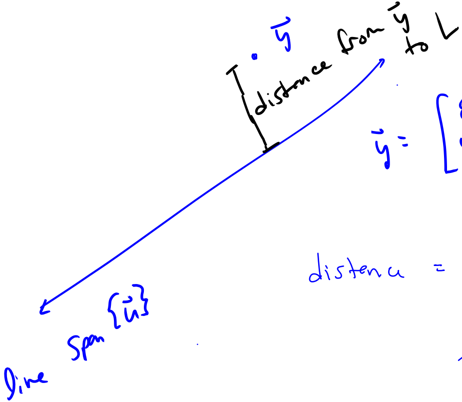
$$\hat{y} = \frac{\vec{y}\cdot\vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{40}{20}\vec{u} = 2\begin{bmatrix} 4 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \end{bmatrix}$$

From $\vec{y} = \hat{y} + \vec{z}$, $\vec{z} = \vec{y} - \hat{y} = \begin{bmatrix} 7 \\ 6 \end{bmatrix} - \begin{bmatrix} 8 \\ 4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

So $\vec{y} = \begin{bmatrix} 8 \\ 4 \end{bmatrix} + \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

Example Continued...

Determine the distance between the point $(7, 6)$ and the line Span$\{\mathbf{u}\}$.



$$\vec{y} = \begin{bmatrix}8\\4\end{bmatrix} + \begin{bmatrix}-1\\2\end{bmatrix}$$

$$\text{distance} = \|\vec{z}\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$