March 2 MATH 1112 sec. 52 Spring 2020

Inverse Trigonometric Functions

Question: If someone asks "what is the sine of $\frac{\pi}{6}$?" we can respond with the answer (from memory or perhaps using a calculator) "$\frac{1}{2}$". What if the question is reversed? What if someone asks

"What angle has a sine value of $\frac{1}{2}$?"

$\frac{\pi}{6}$ is one answer.

$\frac{5\pi}{6}$ is another answer!

Restricting the Domain of $\sin(x)$

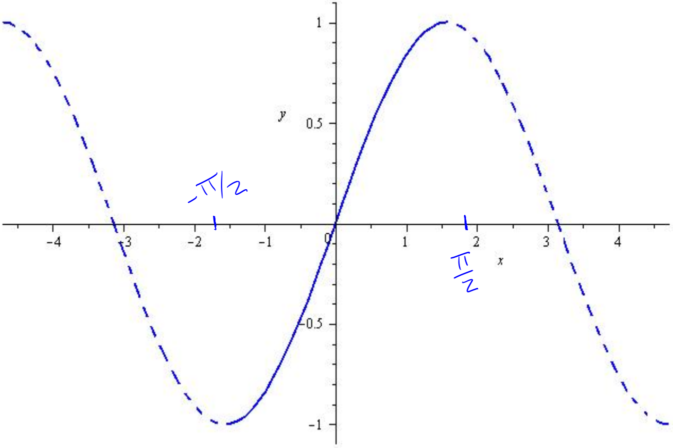


Figure: To define an inverse sine function, we start by restricting the domain of $\sin(x)$ to the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

The Inverse Sine Function (a.k.a. arcsine function)

Definition: For x in the interval $[-1, 1]$ the inverse sine of x is denoted by either

$$\sin^{-1}(x) \quad \text{or} \quad \arcsin(x)$$

and is defined by the relationship

$$y = \sin^{-1}(x) \quad \Longleftrightarrow \quad x = \sin(y) \quad \text{where} \quad -\frac{\pi}{2} \le y \le \frac{\pi}{2}.$$

The Domain of the Inverse Sine is $-1 \le x \le 1$.

The Range of the Inverse Sine is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Notation Warning!

Caution: We must remember not to confuse the superscript -1 notation with reciprocal. That is

$$\sin^{-1}(x) \neq \frac{1}{\sin(x)}.$$

If we want to indicate a reciprocal, we should use parentheses or trigonometric identities

$$\frac{1}{\sin(x)} = (\sin(x))^{-1} \quad \text{or write} \quad \frac{1}{\sin(x)} = \csc(x).$$

Some Inverse Sine Values

We can build a table of some inverse sine values by using our knowledge of the sine function.

x	$\sin(x)$
$-\frac{\pi}{2}$	-1
$-\frac{\pi}{4}$	$-\frac{1}{\sqrt{2}}$
$-\frac{\pi}{6}$	$-\frac{1}{2}$
0	0
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{2}$	1

x	$\sin^{-1}(x)$
-1	$-\frac{\pi}{2}$
$-\frac{1}{\sqrt{2}}$	$-\frac{\pi}{4}$
$-\frac{1}{2}$	$-\frac{\pi}{6}$
0	0
$\frac{1}{\sqrt{2}}$	$\frac{\pi}{4}$
$\frac{\sqrt{3}}{2}$	$\frac{\pi}{3}$
1	$\frac{\pi}{2}$

Conceptual Definition[1]

We can think of the inverse sine function in the following way:

$\sin^{-1}(x)$ **is the *angle* between** $-\frac{\pi}{2}$ **and** $\frac{\pi}{2}$ **whose sine is** x**.**

quadrants IV and I

[1]We want to consider $f(x) = \sin^{-1} x$ as a real valued function of a real variable without necessary reference to angles, triangles, or circles. But the above is a **very useful** conceptual device for working with and evaluating this function.

Example

Evaluate each expression exactly.

(a) $\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$

and $-\frac{\pi}{2} \le \frac{\pi}{4} \le \frac{\pi}{2}$

(b) $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$

$\sin\left(-\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$

and

$-\frac{\pi}{2} \le -\frac{\pi}{3} \le \frac{\pi}{2}$

Question

The value of $\sin^{-1} 1$ is

(a) $\frac{\pi}{2}$

(b) $\frac{\pi}{2}$ and $\frac{5\pi}{2}$

(c) 0

(d) 0 and π

$\sin \frac{\pi}{2} = 1$ and

$-\frac{\pi}{2} \le \frac{\pi}{2} \le \frac{\pi}{2}$

The Graph of the Arcsine

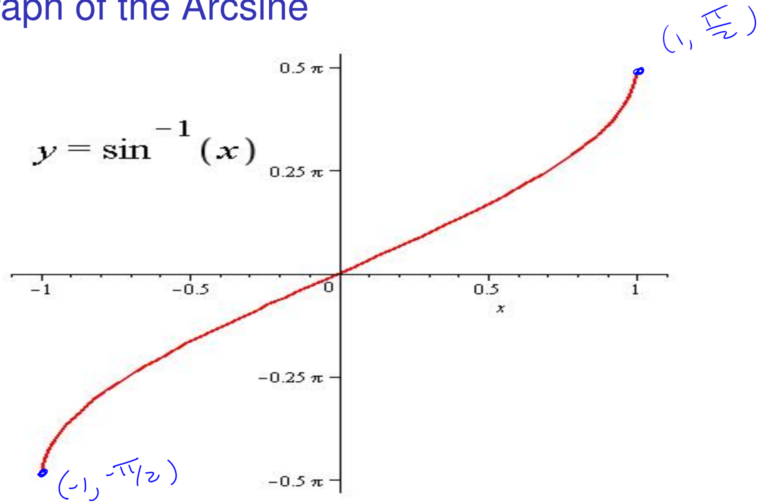


Figure: Note that the domain is $-1 \le x \le 1$ and the range is $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$.

Function/Inverse Function Relationship

For every x in the interval $[-1, 1]$

$$\sin\left(\sin^{-1}(x)\right) = x$$

For every x in the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$\sin^{-1}\left(\sin(x)\right) = x$$

Remark 1: If $x > 1$ or $x < -1$, the expression $\sin^{-1}(x)$ is not defined.

Remark 2: If $x > \frac{\pi}{2}$ or $x < -\frac{\pi}{2}$, the expression $\sin^{-1}\left(\sin(x)\right)$ IS defined, but IS NOT equal to x.

Example

Evaluate each expression if possible. If it is not defined, give a reason.

(a) $\sin\left[\sin^{-1}\left(\frac{1}{2}\right)\right]$ $= \frac{1}{2}$

$\frac{1}{2}$ is in $[-1, 1]$ and for every x in $[-1,1]$

$\sin(\sin^{-1} x) = x$

(b) $\sin^{-1}(3)$

undefined

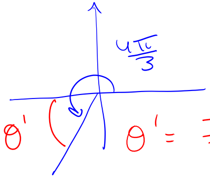
$-1 \leq \sin\theta \leq 1$

What angle between $\frac{-\pi}{2}$ and $\frac{\pi}{2}$ has sine value of 3?

(c) $\sin^{-1}\left[\sin\left(\frac{\pi}{8}\right)\right] = \frac{\pi}{8}$ Since $\frac{\pi}{8}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(d) $\sin^{-1}\left[\sin\left(\frac{4\pi}{3}\right)\right]$

$= \sin^{-1}\left(-\frac{\sqrt{3}}{2}\right) = -\frac{\pi}{3}$



$\theta' = \frac{\pi}{3}$

$\sin\left(\frac{4\pi}{3}\right) = -\sin\frac{\pi}{3}$

$= -\frac{\sqrt{3}}{2}$

Question

The value of $\sin^{-1}\left[\sin \frac{5\pi}{6}\right]$ is

(a) $\frac{5\pi}{6}$

(b) $\frac{\pi}{6}$

(c) $-\frac{\pi}{6}$

(d) $\frac{1}{2}$

(e) $-\frac{1}{2}$

$\frac{5\pi}{6}$ is in Quad II

$$\sin\left(\frac{5\pi}{6}\right) = +\sin\left(\frac{\pi}{6}\right)$$
$$= \sin\left(\frac{\pi}{6}\right)$$