March 2 Math 1190 sec. 63 Spring 2017

Section 3.2: Implicit Differentiation; Derivatives of the Inverse Trigonometric Functions

Implicit differentiation provided the tools necessary to:

- find $\frac{dy}{dx}$ from a relation between x and y,
- extend the power rule to non-integer powers,
- relate derivatives of inverse functions, and
- obtain derivative rules for inverse trigonometric functions.

The Power Rule: Rational Exponents

Theorem: If r is any rational number, then when x^r is defined, the function $y = x^r$ is differentiable and

$$\frac{d}{dx}x^r = rx^{r-1}$$

for all x such that x^{r-1} is defined.

For example, I claimed that $\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}$.

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{1/2} = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2}\frac{1}{\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Example

Let's avoid the quotient rule

Evaluate $\displaystyle \frac{d}{dx}\frac{1+\sqrt{x}}{x^2}$ $= \dfrac{d}{dx}\left(\dfrac{1}{x^2}+\dfrac{\sqrt{x}}{x^2}\right)$

$$= \frac{d}{dx}\left(x^{-2}+\frac{x^{1/2}}{x^2}\right) = \frac{d}{dx}\left(x^{-2}+x^{-3/2}\right)$$

$$= -2x^{-3}-\frac{3}{2}x^{-5/2}$$

$$= \frac{-2}{x^3}-\frac{3}{2x^{5/2}} = \frac{-2}{x^3}\cdot\frac{2}{2}-\frac{3}{2x^{5/2}}\cdot\frac{x^{1/2}}{x^{1/2}}$$

$$= \frac{-4}{2x^3}-\frac{3x^{1/2}}{2x^3} = \frac{-4-3\sqrt{x}}{2x^3}$$

Question

Find $f'(x)$ where $f(x) = \frac{1}{\sqrt{x}}$.

(a) $f'(x) = \frac{1}{2}x^{3/2}$

(b) $f'(x) = -2\sqrt{x}$

(c) $f'(x) = -\frac{2}{x^2}$

(d) $f'(x) = -\frac{1}{2x^{3/2}}$

$f(x) = x^{-1/2}$

$f'(x) = \frac{-1}{2} x^{-1/2 - 1} = \frac{-1}{2} x^{-3/2}$

$= \frac{-1}{2} \frac{1}{x^{3/2}} = \frac{-1}{2x^{3/2}}$

Inverse Functions

Suppose $y = f(x)$ and $x = g(y)$ are inverse functions—i.e. $(g \circ f)(x) = g(f(x)) = x$ for all x in the domain of f.

Theorem: Let f be differentiable on an open interval containing the number x_0. If $f'(x_0) \neq 0$, then g is differentiable at $y_0 = f(x_0)$. Moreover

$$\frac{d}{dy} g(y_0) = g'(y_0) = \frac{1}{f'(x_0)}.$$

Note that this refers to a pair (x_0, y_0) on the graph of f—i.e. (y_0, x_0) on the graph of g. The slope of the curve of f at this point is the reciprocal of the slope of the curve of g at the associated point.

Example: $g'(y_0) = \frac{1}{f'(x_0)}$

We know that $f(x) = e^x$ and $g(x) = \ln x$ are inverse functions. Use this to find $g'(e)$.

we want $g'(e) = \frac{1}{f'(x_0)}$ but we don't know x_0.

So $y_0 = e$. $y_0 = f(x_0) \Rightarrow e = f(x_0) = e^{x_0}$

we need $e^{x_0} = e \Rightarrow x_0 = 1$

$f'(x) = e^x$ so $f'(1) = e^1 = e$

hence $g'(e) = \frac{1}{f'(1)} = \frac{1}{e}$

$g'(y_0) = \frac{1}{f'(x_0)}$

Question

Suppose f and g are inverse functions such that

$$\underbrace{f(2) = 3}, \quad \text{and} \quad f'(2) = 7.$$

$(2,3) = (x_0, y_0)$ $\quad x_0 = 2,\ y_0 = 3$

point on graph of f

$g'(3) = \frac{1}{f'(2)} = \frac{1}{7}$

Then,

(a) $g'(7) = \frac{1}{3}$

(b) $g'(3) = \frac{1}{2}$

(c) $g'(3) = \frac{1}{7}$

(d) $g'(2) = \frac{1}{7}$

Derivative of the Inverse Sine, Tangent, and Secant

We derived (or stated) the three new derivative rules

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \quad |x|<1$$

Recall $y = \sin^{-1}x \Rightarrow x = \sin y$

and $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}} \quad |x|>1$$

The Remaining Inverse Functions

Recall $\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right)$

Due to the trigonometric cofunction identities, it can be shown that

$$\cos^{-1} x = \frac{\pi}{2} - \sin^{-1} x$$

$$\cot^{-1} x = \frac{\pi}{2} - \tan^{-1} x$$

and

$$\csc^{-1} x = \frac{\pi}{2} - \sec^{-1} x$$

Derivatives of Inverse Trig Functions

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}, \qquad \frac{d}{dx}\cos^{-1}x = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}, \qquad \frac{d}{dx}\cot^{-1}x = -\frac{1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}, \qquad \frac{d}{dx}\csc^{-1}x = -\frac{1}{x\sqrt{x^2-1}}$$

Example

Evaluate $\frac{d}{dx}\cot^{-1}\left(\frac{1}{x}\right)$

Inside $u = g(x) = \frac{1}{x} = x^{-1}$

$g'(x) = -1x^{-2} = \frac{-1}{x^2}$

Outside $f(u) = \cot^{-1} u$

$f'(u) = \frac{-1}{1+u^2}$

$$\frac{d}{dx}\cot^{-1}\left(\frac{1}{x}\right) = \frac{-1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2}$$

$$= \frac{1}{\left(1+\frac{1}{x^2}\right)x^2} = \frac{1}{x^2+1}$$

Turns out $\cot^{-1}\left(\frac{1}{x}\right) = \tan^{-1} x$

Question

$$\text{Evaluate} \quad \frac{d}{dx}\csc^{-1}(x^2)$$

(a) $-\dfrac{2x}{x^2\sqrt{x^4-1}}$

(b) $-\dfrac{1}{2x\sqrt{4x^2-1}}$

(c) $-\dfrac{2x}{x\sqrt{x^2-1}}$

(d) $-\dfrac{1}{x^2\sqrt{x^4-1}}$

Inside $u = g(x) = x^2$ $\quad g'(x) = 2x$

outside $f(u) = \csc^{-1} u$

$f'(u) = \dfrac{-1}{u\sqrt{u^2-1}}$

$$\frac{d}{dx}\csc^{-1}(x^2) = \frac{-1}{x^2\sqrt{(x^2)^2-1}} \cdot (2x)$$

Section 3.3: Derivatives of Logarithmic Functions

Recall: If $a > 0$ and $a \neq 1$, we denote the **base** a **logarithm** of x by

$$\log_a x$$

This is the inverse function of the (one to one) function $y = a^x$. So we can define $\log_a x$ by the statement

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

Our present goal is to use our knowledge of the derivative of an exponential function, along with the chain rule, to come up with a derivative rule for logarithmic functions.

Properties of Logarithms

We recall several useful properties of logarithms.

Let a, b, x, y be positive real numbers with $a \neq 1$ and $b \neq 1$, and let r be any real number.

- $\log_a(xy) = \log_a(x) + \log_a(y)$
- $\log_a\left(\frac{x}{y}\right) = \log_a(x) - \log_a(y)$
- $\log_a(x^r) = r\log_a(x)$
- $\log_a(x) = \frac{\log_b(x)}{\log_b(a)}$ (*the change of base formula*)
- $\log_a(1) = 0$

Question

(1) In the expression $\ln(x)$, what is the base?

(a) 10

(b) 1

(c) e

Question

(2) Which of the following expressions is equivalent to

$$\log_2\left(x^3\sqrt{y^2-1}\right)$$

(a) $\log_2(x^3) - \frac{1}{2}\log_2(y^2-1)$

(b) $\frac{3}{2}\log_2(x(y^2-1))$

(c) $3\log_2(x) + \frac{1}{2}\log_2(y^2-1)$

(d) $3\log_2(x) + \frac{1}{2}\log_2(y^2) - \frac{1}{2}\log_2(1)$

$\log_a(x+y) \neq \log_a x + \log_a y$

Properties of Logarithms

Additional properties that are useful.

- $f(x) = \log_a(x)$, has domain $(0, \infty)$ and range $(-\infty, \infty)$.
- For $a > 1$, *

$$\lim_{x\to 0^+} \log_a(x) = -\infty \quad \text{and} \quad \lim_{x\to \infty} \log_a(x) = \infty$$

- For $0 < a < 1$,

$$\lim_{x\to 0^+} \log_a(x) = \infty \quad \text{and} \quad \lim_{x\to \infty} \log_a(x) = -\infty$$

In advanced mathematics (and in light of the change of base formula), we usually restrict our attention to the natural log.

Graphs of Logarithms:Logarithms are continuous on $(0, \infty)$.

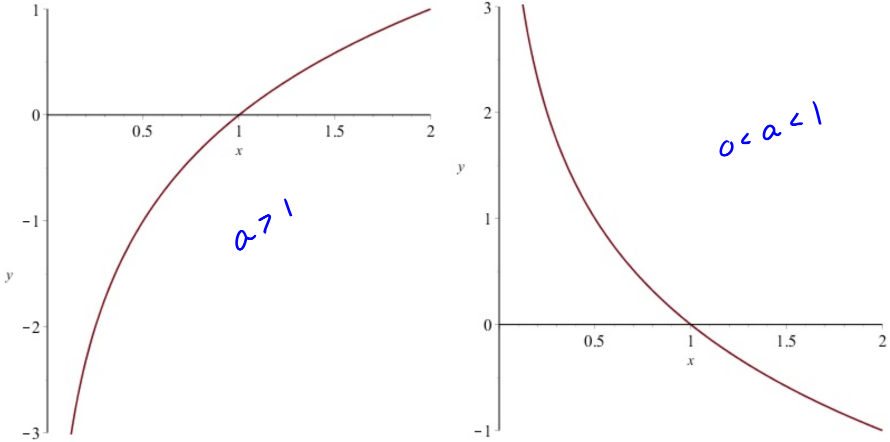
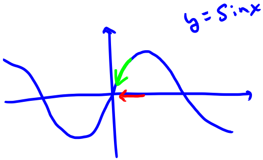


Figure: Plots of functions of the type $f(x) = \log_a(x)$. The value of $a > 1$ on the left, and $0 < a < 1$ on the right.

Examples

Evaluate each limit.

(a) $\lim\limits_{x\to 0^+} \ln(\sin(x)) = -\infty$

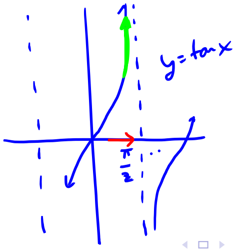


$\lim\limits_{x\to 0^+} \sin x = 0$

from the right

Recall $\lim\limits_{t\to 0^+} \ln t = -\infty$

(b) $\lim\limits_{x\to \frac{\pi}{2}^-} \ln(\tan(x)) = \infty$



$\lim\limits_{x\to \frac{\pi}{2}^-} \tan x = \infty$

$\lim\limits_{t\to \infty} \ln t = \infty$

Question:

$$\lim_{x \to 0^+} \ln x = -\infty, \qquad \lim_{x \to \infty} \ln x = \infty$$

Evaluate the limit $\displaystyle \lim_{x \to \infty} \ln\left(\frac{1}{x^2}\right)$

$$\lim_{x \to \infty} \frac{1}{x^2} = 0$$

through positive numbers

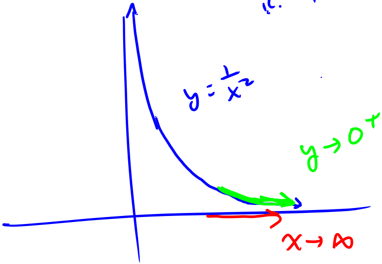
i.e. from the right ↓ above

(a) $-\infty$

(b) 0

(c) ∞

(d) The limit doesn't exist.



Logarithms are Differentiable on Their Domain

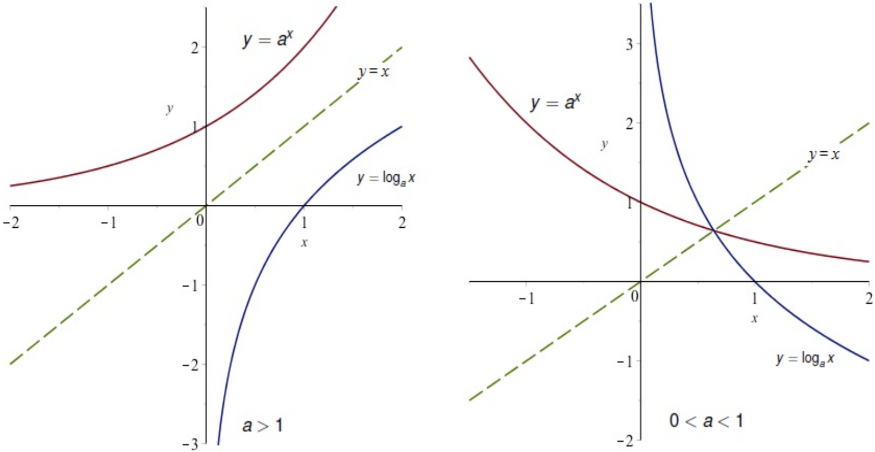


Figure: Recall $f(x) = a^x$ is differentiable on $(-\infty, \infty)$. The graph of $\log_a(x)$ is a reflection of the graph of a^x in the line $y = x$. So $f(x) = \log_a(x)$ is differentiable on $(0, \infty)$.