# March 2 Math 3260 sec. 51 Spring 2020

## Section 4.1: Vector Spaces and Subspaces

**Definition** A **vector space** is a nonempty set $V$ of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in $V$, and for any scalars $c$ and $d$

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in $V$.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in $V$ such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar $c$, $c\mathbf{u}$ is in $V$.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

# Subspaces

**Definition:** A **subspace** of a vector space $V$ is a subset $H$ of $V$ for which

a) The zero vector is in $H$[1]

b) $H$ is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in $H$ implies $\mathbf{u} + \mathbf{v}$ is in $H$)

c) $H$ is closed under scalar multiplication. (i.e. $\mathbf{u}$ in $H$ implies $c\mathbf{u}$ is in $H$)

[1] This is sometimes replaced with the condition that $H$ is nonempty.

# An Example of a Vector Space & Subspace

$C^1(\mathbb{R})$ denotes the set of all real valued functions with domain $\mathbb{R}$ that are one-times continuously differentiable.

A function $f$ is in $C^1(\mathbb{R})$ if

- $f'(x)$ exists, and
- $f'(x)$ is continuous on $(-\infty, \infty)$.

This is a vector space with vector addition and scalar multiplication defined in the standard was for functions:

$$(f+g)(x) = f(x) + g(x), \quad \text{and} \quad (cf)(x) = cf(x).$$

$$H = \{f \in C^1(\mathbb{R}) \mid f(0) = 0\}$$

Show[2] that $H$ is a subspace of $C^1(\mathbb{R})$.

We have to show that the zero vector is in $H$ and $H$ is closed under both operations.

① Note that if $z(x) = 0$ for all real $x$, then $z(0) = 0$. So $H$ contains the zero vector.

② Suppose $f$ and $g$ are in $H$. Then $f(0) = 0$ and $g(0) = 0$. Note

$$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$$

[2] The zero vector in $C^1(\mathbb{R})$ is the function $z(x) = 0$ for all $x$.

So $f+g$ is in $H$, and $H$ is closed under vector addition.

③ Let $f$ be in $H$ and $c$ a scalar. Note

$$(cf)(0) = c\,f(0) = c(0) = 0.$$

So $cf$ is in $H$, and $H$ is closed under scalar multiplication.

$H$ is a subspace of $C^1(\mathbb{R})$.

# Definition: Linear Combination and Span

**Definition** Let $V$ be a vector space and $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$ be a collection of vectors in $V$. A **linear combination** of the vectors is a vector $\mathbf{u}$

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

for some scalars $c_1, c_2, \ldots, c_p$.

**Definition** The **span**, $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is the subet of $V$ consisting of all linear combinations of the vectors $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$.

# Theorem

**Theorem:** If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, for $p \geq 1$, are vectors in a vector space $V$, then Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is a subspace of $V$.

**Remark** This is called the **subspace of $V$ spanned by (or generated by)** $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Moreover, if $H$ is any subspace of $V$, a **spanning set** for $H$ is any set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $H =$Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

# Example

$M^{2\times 2}$ denotes the set of all $2\times 2$ matrices with real entries. Consider the subset $H$ of $M^{2\times 2}$

$$H=\left\{\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}\;\Big|\; a,\, b\in\mathbb{R}\right\}.$$

Show that $H$ is a subspace of $M^{2\times 2}$ by finding a spanning set. That is, show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for some appropriate vectors.

Let $\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}$ be any element of $H$.

We'll write this as a linear combination.

$$\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix} = \begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix} 0 & 0\\ 0 & b\end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This is a linear combo of the two vectors $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

So $H = \operatorname{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.

As a span of some set, $H$ is a subspace of $M^{2\times 2}$.

# Section 4.2: Null & Column Spaces, Linear Transformations

**Definition:** Let $A$ be an $m \times n$ matrix. The **null space** of $A$, denoted[3] by Nul $A$, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

We can say that Nul $A$ is the subset of $\mathbb{R}^n$ that gets mapped to the zero vector under the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

[3]Some authors will write Null($A$) with two ells.

# Example

Determine Nul $A$ where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}.$$

We have to characterize all vectors $\vec{x}$ in $\mathbb{R}^3$ such that $A\vec{x} = \vec{0}$. We can use an augmented matrix $[A\ \vec{0}]$.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 7 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$x_1 = -3x_3$

$x_2 = -2x_3$

$x_3$ is free

$$A\vec{x} = \vec{0} \quad \Rightarrow \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}.$$