March 2 Math 3260 sec. 55 Spring 2020

Section 4.1: Vector Spaces and Subspaces

Definition A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Subspaces

Definition: A **subspace** of a vector space V is a subset H of V for which

a) The zero vector is in H[1]

b) H is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in H implies $\mathbf{u} + \mathbf{v}$ is in H)

c) H is closed under scalar multiplication. (i.e. $\mathbf{u}$ in H implies $c\mathbf{u}$ is in H)

[1] This is sometimes replaced with the condition that H is nonempty.

An Example of a Vector Space & Subspace

$C^1(\mathbb{R})$ denotes the set of all real valued functions with domain $\mathbb{R}$ that are one-times continuously differentiable.

A function f is in $C^1(\mathbb{R})$ if

- $f'(x)$ exists, and
- $f'(x)$ is continuous on $(-\infty, \infty)$.

This is a vector space with vector addition and scalar multiplication defined in the standard was for functions:

$$(f+g)(x) = f(x) + g(x), \quad \text{and} \quad (cf)(x) = cf(x).$$

$$H = \{f \in C^1(\mathbb{R}) \mid f(0) = 0\}$$

Show[2] that H is a subspace of $C^1(\mathbb{R})$.

We need to show that H has the three properties of a subspace.

① Note that if $z(x) = 0$ for all x, then $z(0) = 0$. So the set contains the zero vector.

② Suppose f and g are in H. Then $f(0) = 0$ and $g(0) = 0$.

$$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$$

Thus $f+g$ is in H, and H is closed

[2] The zero vector in $C^1(\mathbb{R})$ is the function $z(x) = 0$ for all x.

under vector addition.

(3) Let f be in H, and c be a scalar.

Then $(cf)(0) = cf(0) = c \cdot 0 = 0$

So cf is in H, and H is closed under scalar multiplication.

H is a subspace of $C^1(\mathbb{R})$.

Definition: Linear Combination and Span

Definition Let V be a vector space and $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$ be a collection of vectors in V. A **linear combination** of the vectors is a vector $\mathbf{u}$

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

for some scalars $c_1, c_2, \ldots, c_p$.

Definition The **span**, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is the subet of V consisting of all linear combinations of the vectors $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$.

Theorem

Theorem: If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, for $p \geq 1$, are vectors in a vector space V, then Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is a subspace of V.

Remark This is called the **subspace of V spanned by (or generated by)** $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Moreover, if H is any subspace of V, a **spanning set** for H is any set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $H =$Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example

$M^{2\times 2}$ denotes the set of all 2×2 matrices with real entries. Consider the subset H of $M^{2\times 2}$

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Show that H is a subspace of $M^{2\times 2}$ by finding a spanning set. That is, show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for some appropriate vectors.

We'll take an element of H and write it as a linear combination.

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

$$= a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

This is a linear combo of the vectors

$$\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \text{ and } \vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$H = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$

Section 4.2: Null & Column Spaces, Linear Transformations

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted[3] by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

We can say that Nul A is the subset of $\mathbb{R}^n$ that gets mapped to the zero vector under the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

[3] Some authors will write Null(A) with two ells.

Example

Determine Nul A where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}.$$

We need to characterize vectors $\vec{x}$ in $\mathbb{R}^3$ such that $A\vec{x} = \vec{0}$. We can use row reduction on $[A\ \vec{0}]$.

$$\begin{bmatrix} 1 & 0 & 3 & 0 \\ 1 & 2 & 7 & 0 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 2 & 0 \end{bmatrix}$$

$x_1 = -3x_3$

$x_2 = -2x_3$

x_3 - free

A vector $\vec{x}$ in Nul A is

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Theorem

For $m \times n$ matrix A, Nul A is a subspace of $\mathbb{R}^n$.