March 30 Math 1190 sec. 62 Spring 2017

Section 5.1: Area (under the graph of a nonnegative function)

We're solving the problem of finding the area enclosed between the graph of a function f and the x-axis on the interval $[a, b]$ under the assumptions that

- f is continuous on the interval $[a, b]$, and
- f is nonnegative, i.e $f(x) \geq 0$, on $[a, b]$.

Area as the Limit of Riemann Sums

- We made a partition $a = x_0 < x_1 < \cdots < x_n = b$,
- approximated the area of each piece with a rectangle of height $f(c_i)$ and width Δx
- approximate the whole area with the sum of the areas of the rectangles
$$A \approx \sum_{i=1}^{n} f(c_i)\Delta x$$
- then the true area is given by the limit
$$A = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x$$

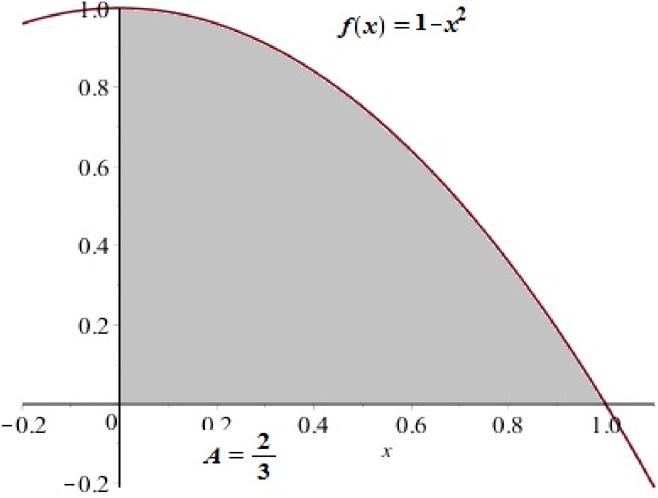


Figure: We found the area under the curve $f(x) = 1 - x^2$ over the interval $[0, 1]$. The area was $\frac{2}{3}$.

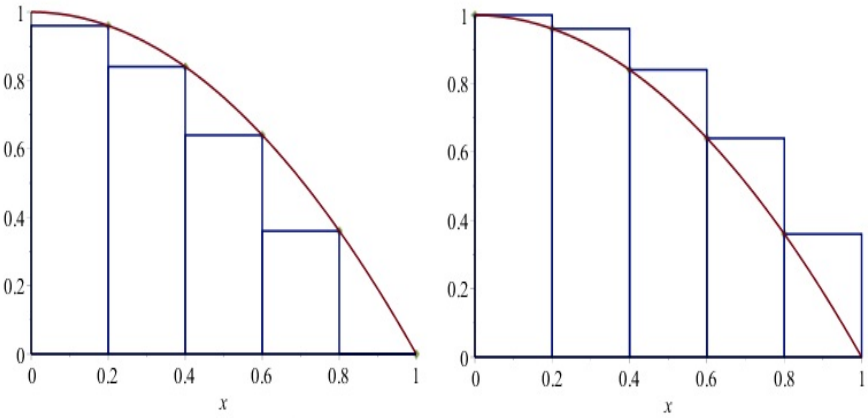


Figure: We can use right or left end points to define the rectangle heights.

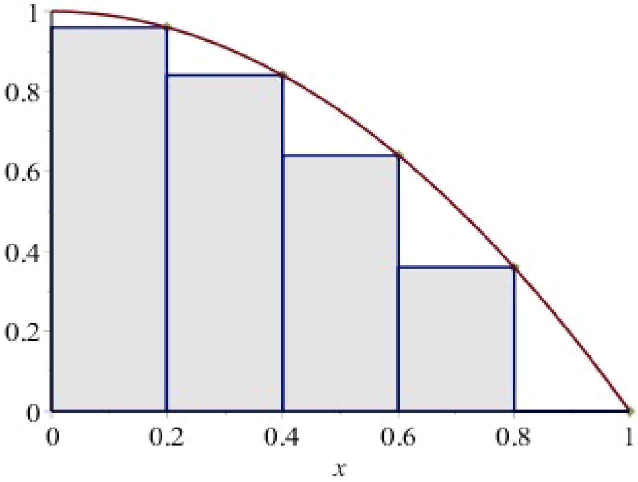


Figure: Here's the region with 5 rectangles using right end points. $A \approx \frac{14}{25}$

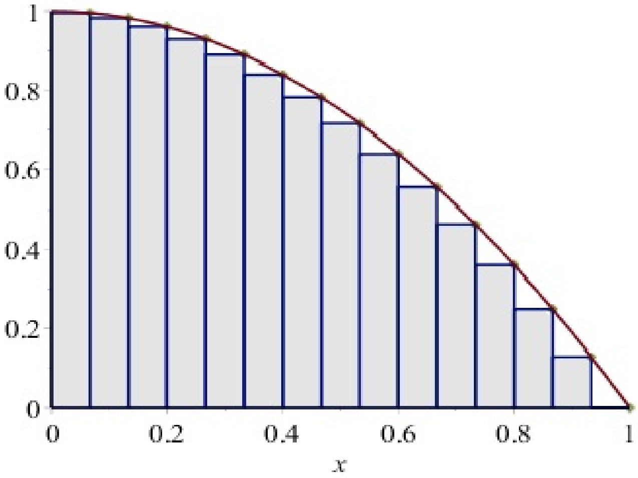


Figure: Here's the region with 15 rectangles using right end points. $A \approx \frac{427}{675}$ (for reference, the true area is $450/675$)

Riemann Sum Demo

GeoGebra Riemann Sum Demo

Equally Spaced Partition Case:

- $\Delta x = \frac{b-a}{n}$
- $x_0 = a$, $x_1 = a + \Delta x$, $x_2 = a + 2\Delta x$, i.e. $x_i = a + i\Delta x$
- Taking heights to be

$$\text{left ends} \quad c_i = x_{i-1} \quad \text{area} \approx \sum_{i=1}^{n} f(x_{i-1})\Delta x$$

$$\text{right ends} \quad c_i = x_i \quad \text{area} \approx \sum_{i=1}^{n} f(x_i)\Delta x$$

- The true area exists (for f continuous) and is given by

$$\lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x.$$

A sum of the form $\sum_{i=1}^{n} f(x_i)\Delta x$ is called a **Riemann sum**.

An Example: Recovering Distance from Velocity

The speedometer readings for a motorcycle are recorded at 12 second intervals. Use the information in the table to estimate the total distance traveled. Get estimates using

(a) left end points (beginning time of intervals), and
(b) right end points (ending time for each interval).

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

Distance = rate × time

note the form like height × width

- left ends
- right ends

Approximate by assuming velocity is constant on each interval

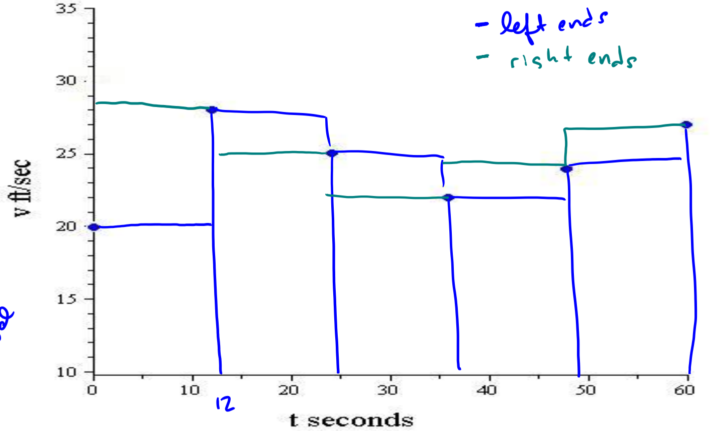


Figure: Graphical representation of motorcycle's velocity.

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

left ends

$$D \approx 20\frac{ft}{s}\cdot 12s + 28\frac{ft}{s}\cdot 12s + 25\frac{ft}{s}\cdot 12s + 22\frac{ft}{s}\cdot 12s + 24\frac{ft}{sec}\cdot 12s$$

$$= 1428\ ft$$

We're recovering distance related to area.

Note also Distance is a displacement (total change in position)

Velocity is rate of change of position.

t in sec	0	12	24	36	48	60
v in ft/sec	20	28	25	22	24	27

Right ends

$$D \approx 28\,\frac{ft}{s}\cdot 12\,s + 25\,\frac{ft}{s}\cdot 12\,s + 22\,\frac{ft}{s}\cdot 12\,s$$

$$+ 24\,\frac{ft}{s}\cdot 12\,s + 27\,\frac{ft}{s}\cdot 12\,s$$

$$= 1512 \text{ ft}$$

Our Motorcycle's True Velocity is Probably "Smooth"

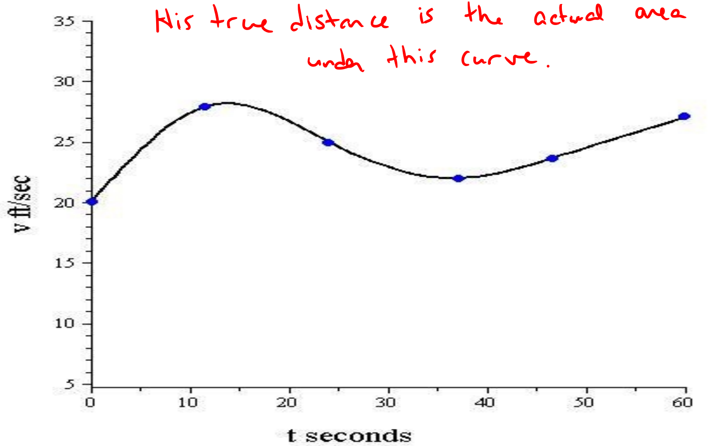


Figure: The true graph of the velocity probably looks more like this. But we only know for certain what it is at the recorded times.

Section 5.2: The Definite Integral

We saw that a sum of the form

$$f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$$

approximated the area of a region if f was continuous and positive. And that under these conditions, the limit

$$\lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta x = \lim_{n\to\infty}[f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x]$$

was the value of this area.

Can we generalize this dropping the requirement that f is positive? that f is continuous?

Definition (Definite Integral)

Let f be defined on an interval $[a, b]$. Let

$$x_0 = a < x_1 < x_2 < \cdots < x_n = b$$

be any partition of $[a, b]$, and $\{c_1, c_2, \ldots, c_n\}$ be any set of sample points. Then the **definite integral of** f **from** a **to** b is denoted and defined by

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(c_i)\Delta x_i$$

provided this limit exists. Here, the limit is taken over all possible partitions of $[a, b]$.

Terms and Notation

- **Riemann Sum:** any sum of the form $f(c_1)\Delta x + f(c_2)\Delta x + \cdots + f(c_n)\Delta x$
- **Integral Symbol/Sign:** $\int$ (a stretched "S" for "sum")
- **Integrable:** If the limit does exists, f is said to be integrable on $[a, b]$
- **Limits of Integration:** a is called the lower limit of integration, and b is the upper limit of integration
- **Integrand:** the expression "$f(x)$" is called the integrand

- **Differential:** dx is called a differential, it indicates what the variable is and can be thought of as the limit of Δx (just as it is in the derivative notation "$\frac{dy}{dx}$").

- **Dummy Variable/Variable of Integration:** the variable that appears in both the integrand and in the differential. For example, if the differential is dx, the dummy variable is x; it the differential is du, the dummy variable is u

$$\int_a^b f(x)\,dx$$

Important Remarks

(1) If the integral does exist, it is a **number**. That is, it does not depend on the dummy variable of integration. The following are equivalent.

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(q)\,dq$$

(2) The definite integral is a limit of Riemann Sums!

e.g. $\int_0^1 (1-x^2)\,dx = 2/3$

(3) If f is positive and continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{ the area under the curve.}$$

Questions

Consider the integral $\int_{-3}^{\pi} f(r)\,dr$

(1) The dummy variable of integration is

(a) x
(b) r
(c) can't be determined without more information
(d) dr

Questions

(2) If it is known that $\int_{-3}^{\pi} f(r)\,dr = 7$, then

$$\int_{-3}^{\pi} f(x)\,dx = \int_{-3}^{\pi} f(r)\,dr = \int_{-3}^{\pi} f(☺)\,d☺$$

(a) $7x$

(b) -7

(c) can't be determined without more information

(d) 7

What if f is continuous, but not always positive?

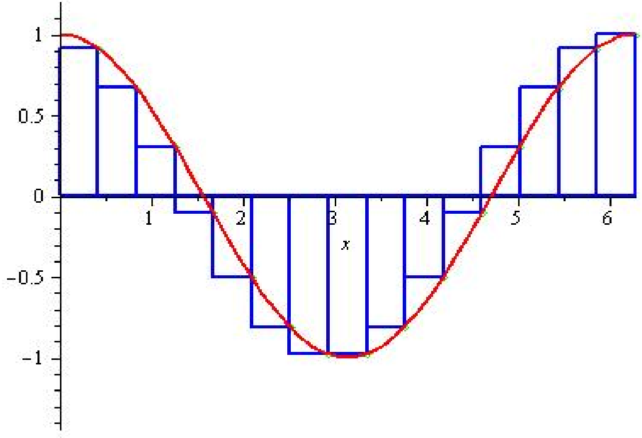


Figure: A function that changes signs on $[a, b]$. (Here, $f(x) = \cos x$, $a = 0$ and $b = 2\pi$; the partition has 15 subintervals.)

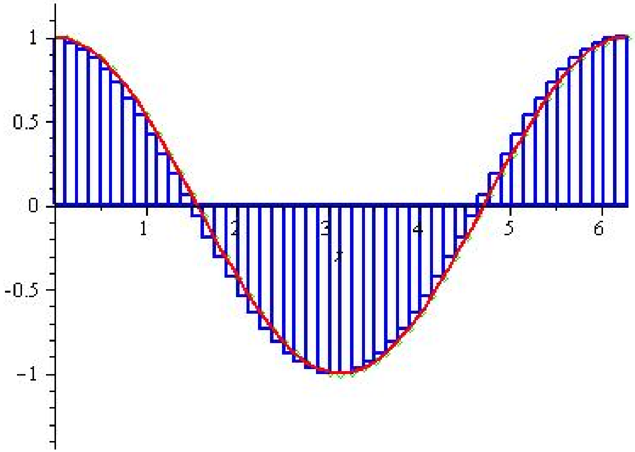


Figure: The same function but with 50 subintervals.

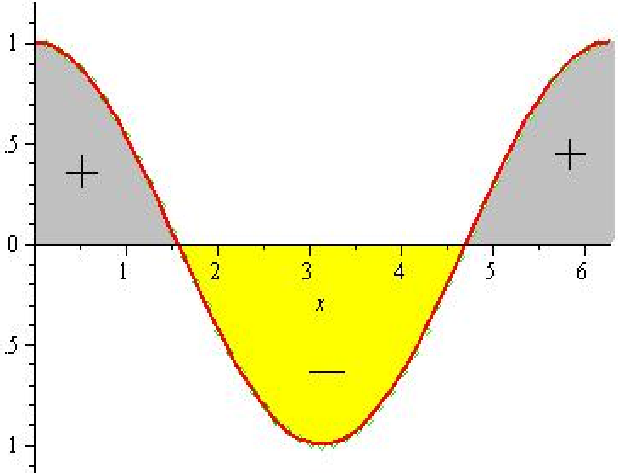
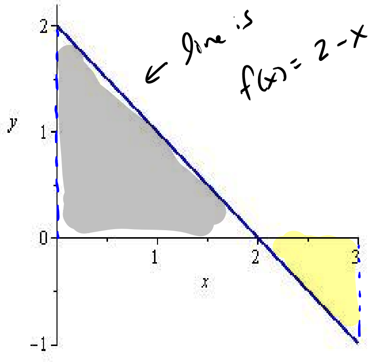


Figure: $\int_a^b f(x)\,dx$ = area of gray region − area of yellow region

Example

Use area to evaluate the integral $\int_0^3 (2-x)\,dx$.



$\int_0^3 (2-x)\,dx$ = Gray area minus Yellow area

Gray: Triangle base = 2 height = 2

Area = $\frac{1}{2}bh = \frac{1}{2}(2)(2) = 2$

Yellow : Triangle base = 1 height = 1

$$\text{Area} = \frac{1}{2}bh = \frac{1}{2}(1)(1) = \frac{1}{2}$$

So $$\int_0^3 (2-x)\,dx = 2 - \frac{1}{2} = \frac{3}{2}$$