Mar. 30 Math 2254H sec 015H Spring 2015

Section 11.5: Alternating Series

Definition: Let $\{b_n\}$ be a sequence of nonnegative numbers. A series of the form

$$\sum_{n=1}^{\infty}(-1)^n b_n, \quad \text{or} \quad \sum_{n=1}^{\infty}(-1)^{n-1} b_n$$

is called an **alternating series.**

$$\sum_{n=1}^{\infty}(-1)^n b_n = -b_1 + b_2 - b_3 + b_4 - \cdots$$

$$\sum_{n=1}^{\infty}(-1)^{n-1} b_n = b_1 - b_2 + b_3 - b_4 + \cdots$$

Examples of Alternating Series

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is called the **alternating harmonic series.**

The series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n}\, n}{n+2} = -\frac{1}{3} + \frac{2}{4} - \frac{3}{5} + \frac{4}{6} - \cdots$$

is an alternating series.

Theorem: The Alternating Series Test

Theorem: Let $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ be an alternating series. If

$$(i) \quad b_{n+1} \leq b_n \qquad \text{for all } n$$

$$\text{and} \quad (ii) \quad \lim_{n\to\infty} b_n = 0,$$

Then the series is convergent.

Example

(a) Determine the convergence or divergence of the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$.

Use Alt. Series test: $b_n = \frac{1}{n}$

(i) $\frac{1}{n+1} \leq \frac{1}{n}$ for all $n \geq 1$

so $b_{n+1} \leq b_n$

(ii) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$

The series is convergent.

Example

Determine the convergence or divergence of the series

(b) $$\sum_{n=1}^{\infty} \frac{(-1)^n n}{n+2}$$

Alt. Series test:

$$b_n = \frac{n}{n+2}$$

$$\text{(i)} \quad \lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n+2}$$

$$= \lim_{n\to\infty} \frac{n}{n+2} \cdot \frac{\frac{1}{n}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{1}{1+\frac{2}{n}} = 1$$

$$\lim_{n\to\infty} \frac{(-1)^n n}{n+2} \quad \text{DNE}$$

even terms tend to 1

odd terms tend to -1

The series diverges by the divergence test.

Example

Determine the convergence or divergence of the series

(c) $\sum_{n=2}^{\infty}(-1)^{n+1}\frac{n}{n^2-3}$

Alt Series test.

$b_n = \frac{n}{n^2-3} \quad n \geq 2$

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2-3}$

$= \lim_{n\to\infty} \frac{n}{n^2-3}\left(\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right) = \lim_{n\to\infty} \frac{\frac{1}{n}}{1-3/n^2} = 0$

Condition (i) holds.

i) Let $f(x) = \dfrac{x}{x^2-3}$ so $f(n) = b_n$

$$f'(x) = \frac{x^2-3-(2x)x}{(x^2-3)^2} = \frac{-(x^2+3)}{(x^2-3)^2} < 0 \quad \text{for all } x \geq 2$$

So $b_{n+1} = f(n+1) \leq f(n) = b_n$

Condition i) also holds.

The series is convergent.

Example

Determine the convergence or divergence of the series

(d) $\sum_{n=1}^{\infty} \cos(n\pi)\left(1+\frac{1}{n}\right)^n$

$\cos(n\pi) = \begin{cases} 1, & n\text{ even} \\ -1, & n\text{-odd} \end{cases} = (-1)^n$

$= \sum_{n=1}^{\infty} (-1)^n \left(1+\frac{1}{n}\right)^n$

$b_n = \left(1+\frac{1}{n}\right)^n$

Alt. Series test:

(i) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

$$\lim_{n\to\infty} (-1)^n \left(1+\frac{1}{n}\right)^n \text{ DNE}$$

This series diverges by the divergence test.

An Observation

Note: If property (ii) doesn't hold, i.e. if $\lim_{n\to\infty} b_n \neq 0$, then the series will definitely diverge by the divergence test.

However, if the second condition DOES hold, but the first does not, the test is inconclusive. The series may converge or it may diverge. Some other test must be used.

A Strange Case: ($b_{n+1} \leq b_n$ is required)

Consider the series

$$\sum_{n=1}^{\infty} (-1)^n b_n \quad \text{where} \quad b_n = \begin{cases} \frac{4}{(n+1)^2}, & n \text{ odd} \\ \frac{2}{n}, & n \text{ even} \end{cases}$$

It is easy to see that $\lim_{n\to\infty} b_n = 0$. But note that the terms b_n are

$$\{b_n\} = \left\{1, 1, \frac{1}{4}, \frac{1}{2}, \frac{1}{9}, \frac{1}{3}, \frac{1}{16}, \frac{1}{4}, \ldots\right\}$$

So that

$$\sum_{n=1}^{\infty} (-1)^n b_n = \sum_{n=1}^{\infty} \frac{1}{n} - \sum_{n=1}^{\infty} \frac{1}{n^2}$$

which is **divergent.**

Another Strange Case:

Consider the series

$$\sum_{n=1}^{\infty}(-1)^n b_n \quad \text{where} \quad b_n = \begin{cases} \frac{4}{(n+1)^2}, & n \text{ odd} \\ \frac{8}{n^3}, & n \text{ even} \end{cases}$$

It is easy to see that $\lim_{n\to\infty} b_n = 0$. But note that the terms b_n are

$$\{b_n\} = \left\{1, 1, \frac{1}{4}, \frac{1}{8}, \frac{1}{9}, \frac{1}{27}, \frac{1}{16}, \frac{1}{64}, \frac{1}{25}, \frac{1}{125}, \frac{1}{36}, \ldots\right\}$$

So that

$$\sum_{n=1}^{\infty}(-1)^n b_n = \sum_{n=1}^{\infty}\frac{1}{n^3} - \sum_{n=1}^{\infty}\frac{1}{n^2}$$

which is **convergent**.

Section 11.6: Absolute Convergence & the Ratio Test

Note that

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \quad \text{converges,}$$

but

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n}\right| = \sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots \quad \text{diverges.}$$

However, both

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \cdots, \quad \text{and}$$

$$\sum_{n=1}^{\infty}\left|(-1)^{n-1}\frac{1}{n^2}\right| = \sum_{n=1}^{\infty}\frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots \quad \text{converge.}$$

Absolute Convergence

Definition: A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

For Example: The series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2} \quad \text{is } \textbf{absolutely convergent.}$$

The alternating harmonic series is **NOT** absolutely convergent.

Conditional Convergence

Definition: A series that is convergent but is not absolutely convergent is called **conditionally convergent**.

The alternating harmonic series IS conditionally convergent.

Theorem on Absolute Convergence

Theorem: If a series is absolutely convergent, it is convergent.

Remark: If we can show that a series is absolutely convergent, then we can conclude that it is convergent.

Remark: Of course, this doesn't mean that a series that isn't absolutely convergent must diverge. It may be conditionally convergent, and some effort may be required to determine its nature.

Example

Determine if the series is convergent or divergent.

$$\sum_{n=1}^{\infty} \frac{\sin n}{n^3}$$

Consider $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^3} \right|$ a series of positive terms

Direct comparison test:

$$0 \leq \frac{|\sin(n)|}{n^3} \leq \frac{1}{n^3}$$

$\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges (p-series) so $\displaystyle\sum_{n=1}^{\infty} \left| \frac{\sin(n)}{n^3} \right|$ converges. Hence $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(n)}{n^3}$ is absolutely convergent.

Theorem: The Ratio Test (a test for abs. convergence)

Theorem: Let $\sum a_n$ be a series, and define the number L by

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

If

(i) $L < 1$, the series is absolutely convergent;

(ii) $L > 1$, the series is divergent;

(iii) $L = 1$, the test is inconclusive.

Remark: In the case $L = 1$, the series may be absolutely convergent, conditionally convergent, or divergent. This test truly **fails**, and some other test or analysis is necessary to draw any conclusion.

Examples

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(a) $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^2}{4^n}$

Try the ratio test : $a_n = \dfrac{(-1)^n n^2}{4^n}$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)^2}{4^{n+1}} \div \frac{(-1)^n n^2}{4^n}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{(-1)^{n+1}}(n+1)^2\,\cancel{4^n}}{\cancel{4^{n+1}}\,\cancel{(-1)^n}\,n^2}\right| = \lim_{n\to\infty}\frac{1}{4}\frac{(n+1)^2}{n^2}$$

$$= \lim_{n\to\infty}\frac{1}{4}\left(\frac{n+1}{n}\right)^2$$

$$= \lim_{n\to\infty} \frac{1}{4}\left(1+\frac{1}{n}\right)^2 = \frac{1}{4}(1)^2 = \frac{1}{4}$$

$$L = \frac{1}{4} < 1$$

The series is absolutely convergent.