Mar. 31 Math 2254H sec 015H Spring 2015

Section 11.6: Absolute Convergence: the Ratio & Root Tests

Definition: A series $\sum a_n$ is called **absolutely convergent** if the series of absolute values $\sum |a_n|$ is convergent.

Theorem: If $\sum a_n$ is absolutely convergent, it is convergent.

Definition: If $\sum a_n$ is convergent, and $\sum |a_n|$ is divergent, then $\sum a_n$ is called **conditionally convergent.**

Conditionally Convergent Series

Suppose $\sum a_n$ is conditionally convergent. If R is any real number, then there is a rearrangement of the terms a_n that sums to R.

Example: It can be shown that as written $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$. Consider rearranging the terms with the pattern

One positive, next two negatives, next one positive, next two negatives, etc.

$$\ln(2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots$$

$$= 1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \frac{1}{7} - \frac{1}{14} - \frac{1}{16} + \frac{1}{9} - \cdots$$

$$= \left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) - \frac{1}{12} + \left(\frac{1}{7} - \frac{1}{14}\right) - \frac{1}{16} + \cdots$$

$$= \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots$$

$$= \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots\right.$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}\right)$$

$$= \frac{1}{2} \ln 2 \quad ?$$

The sum is "conditional." It depends on the arrangement!

Theorem: The Ratio Test (a test for abs. convergence)

Theorem: Let $\sum a_n$ be a series, and define the number L by

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L.$$

If

(i) $L < 1$, the series is absolutely convergent;

(ii) $L > 1$, the series is divergent;

(iii) $L = 1$, the test is inconclusive.

Remark: In the case $L = 1$, the series may be absolutely convergent, conditionally convergent, or divergent. This test truly **fails**, and some other test or analysis is necessary to draw any conclusion.

Examples

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(b) $\sum_{n=1}^{\infty} \frac{n^n}{n!}$

Ratio test: $a_n = \frac{n^n}{n!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}}{(n+1)!} \div \frac{n^n}{n!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^{n+1}\, n!}{n^n (n+1)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^n \cancel{(n+1)}\ \cancel{n!}}{n^n\ \cancel{n!}\ \cancel{(n+1)}} \right|$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e \qquad L = e \text{ here.}$$

$e > \underline{1}$. So this series is divergent.

Examples

Determine if the series is absolutely convergent, conditionally convergent, or divergent.

(c) $\sum_{n=0}^{\infty}(-1)^n \frac{\pi^{2n}}{(2n)!}$

Ratio test $a_n = \frac{(-1)^n \pi^{2n}}{(2n)!}$

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(-1)^{n+1} \pi^{2(n+1)}}{(2(n+1))!} \cdot \frac{(2n)!}{(-1)^n \pi^{2n}} \right|$$

$$= \lim_{n\to\infty} \frac{\pi^{\cancel{2n}+2}}{(2n+2)!} \cdot \frac{(2n)!}{\cancel{\pi^{2n}}}$$

$$= \lim_{n\to\infty} \frac{\pi^2 \cancel{(2n)!}}{\cancel{(2n)!}(2n+1)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{\pi^2}{(2n+1)(2n+2)} = 0$$

$L = 0 < 1$

The series is absolutely convergent.