March 31 Math 2306 sec 58 Spring 2016

Section 11: Linear Mechanical Equations

We can consider the application of an external driving force (with or without damping). Assume a time dependent force $f(t)$ is applied to the system. The ODE governing displacement becomes

$$m\frac{d^2x}{dt^2} = -\beta\frac{dx}{dt} - kx + f(t), \quad \beta \ge 0.$$

Divide out m and let $F(t) = f(t)/m$ to obtain the nonhomogeneous equation

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t)$$

Forced Undamped Motion and Resonance

Consider the case $F(t) = F_0 \cos(\gamma t)$ or $F(t) = F_0 \sin(\gamma t)$, and $\lambda = 0$. Two cases arise

$$(1) \quad \gamma \neq \omega, \quad \text{and (2)} \quad \gamma = \omega.$$

Taking the sine case, the DE is

$$x'' + \omega^2 x = F_0 \sin(\gamma t)$$

with complementary solution

$$x_c = c_1 \cos(\omega t) + c_2 \sin(\omega t).$$

If $\gamma \neq \omega$, then $x_p = A\cos(\gamma t) + B\sin(\gamma t)$.
If $\gamma = \omega$, then $x_p = At\cos(\omega t) + Bt\sin(\omega t)$.

Forced Undamped Motion and Resonance

For $F(t) = F_0 \sin(\gamma t)$ starting from rest at equilibrium:

$$\text{Case (1):} \quad x'' + \omega^2 x = F_0 \sin(\gamma t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{\omega^2 - \gamma^2}\left(\sin(\gamma t) - \frac{\gamma}{\omega}\sin(\omega t)\right)$$

If $\gamma \approx \omega$, the amplitude of motion could be rather large!

Pure Resonance

$$\text{Case (2):} \quad x'' + \omega^2 x = F_0 \sin(\omega t), \quad x(0) = 0, \quad x'(0) = 0$$

$$x(t) = \frac{F_0}{2\omega^2}\sin(\omega t) - \frac{F_0}{2\omega} t \cos(\omega t)$$

Note that the amplitude, α, of the second term is a function of t:

$$\alpha(t) = \frac{F_0 t}{2\omega}$$

which grows without bound!

▸ Forced Motion and Resonance Applet

Choose "Elongation diagram" to see a plot of displacement. Try exciter frequencies close to ω.

Section 12: LRC Series Circuits

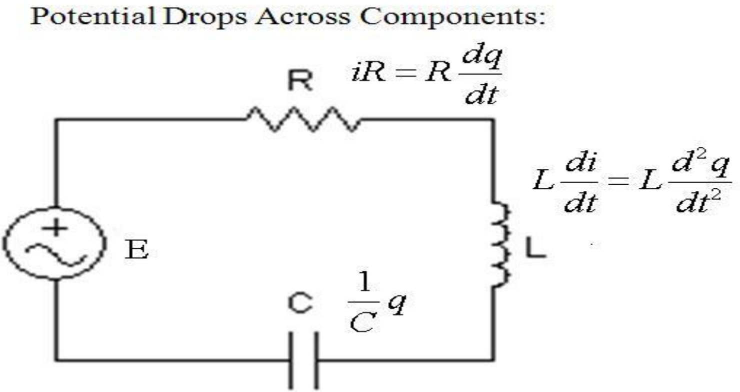

Figure: Kirchhoff's Law: The charge q on the capacitor satisfies $Lq'' + Rq' + \frac{1}{C}q = E(t)$.

LRC Series Circuit (Free Electrical Vibrations)

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0$$

If the applied force $E(t) = 0$, then the **electrical vibrations** of the circuit are said to be **free.** These are categorized as

overdamped if	$R^2 - 4L/C > 0,$ ← 2 real. roots
critically damped if	$R^2 - 4L/C = 0,$ ← one repeated root
underdamped if	$R^2 - 4L/C < 0.$ ↑ complex roots

Steady and Transient States

Given a nonzero applied voltage $E(t)$, we obtain an IVP with nonhomogeneous ODE for the charge q

$$Lq'' + Rq' + \frac{1}{C}q = E(t), \quad q(0) = q_0, \quad q'(0) = i_0.$$

From our basic theory of linear equations we know that the solution will take the form

$$q(t) = q_c(t) + q_p(t).$$

The function of q_c is influenced by the initial state (q_0 and i_0) and will decay exponentially as $t \to \infty$. Hence q_c is called the **transient state charge** of the system.

Steady and Transient States

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$$q(t) = q_c(t) + q_p(t).$$

The function q_p is independent of the initial state but depends on the characteristics of the circuit (L, R, and C) and the applied voltage E. q_p is called the **steady state charge** of the system.

Example

An LRC series circuit has inductance 0.5 h, resistance 10 ohms, and capacitance $4 \cdot 10^{-3}$ f. Find the steady state current of the system if the applied force is $E(t) = 5\cos(10t)$.

Note current = derivative of charge

Eqn. looks like $Lq'' + Rq' + \frac{1}{C} q = E$

Here $L = 0.5$, $R = 10$, $C = 4 \cdot 10^{-3}$, $E(t) = 5\cos(10t)$

So $\frac{1}{2} q'' + 10 q' + \frac{1}{4 \cdot 10^{-3}} q = 5\cos(10t)$

$$2 \cdot \frac{1}{4 \cdot 10^{-3}} = \frac{2 \cdot 10^3}{4} = \frac{10^3}{2} = \frac{1000}{2} = 500$$

Standard form $\quad q'' + 20q' + 500q = 10\cos(10t)$

Char. eqn $\quad r^2 + 20r + 500 = 0$

Complete the square $\qquad 100 = \left(\frac{20}{2}\right)^2$

$$r^2 + 20r + 100 - 100 + 500 = 0$$

$$(r+10)^2 + 400 = 0 \implies (r+10)^2 = -400$$

$$r + 10 = \pm\sqrt{-400} = \pm 20i$$

$$r = -10 \pm 20i$$

$$q_1 = e^{-10t}\cos(20t), \quad q_2 = e^{-10t}\sin(20t)$$

$$q'' + 20q' + 500q = 10\cos(10t)$$

we need q_p: Guess $q_p = A\cos(10t) + B\sin(10t)$

$$q_p' = -10A\sin(10t) + 10B\cos(10t)$$

$$q_p'' = -100A\cos(10t) - 100B\sin(10t)$$

$$q_p'' + 20q_p' + 500q_p =$$

$$-100A\cos(10t) - 100B\sin(10t) - 200A\sin(10t) + 200B\cos(10t) +$$

$$500A\cos(10t) + 500B\sin(10t) = 10\cos(10t) + 0\sin(10t)$$

Collect Cosines + Sines

$$\cos(10t)\left[-100A+200B+500A\right]+\sin(10t)\left[-100B-200A+500B\right]$$

$$= 10\cos(10t) + 0\cdot\sin(10t)$$

$$400A + 200B = 10$$
$$-200A + 400B = 0$$

$$400A + 200B = 10$$
$$-400A + 800B = 0$$

add

$$1000B = 10$$
$$B = \frac{1}{100}$$

$$200A = 400B \implies A = 2B = \frac{2}{100} = \frac{1}{50}$$

So the steady state charge is

$$q_p = \frac{1}{50}\cos(10t) + \frac{1}{100}\sin(10t)$$

The steady state current

is

$$i_p = \frac{-1}{5}\sin(10t) + \frac{1}{10}\cos(10t)$$

$\left(i = \frac{dq}{dt} \right)$

Section 13: The Laplace Transform

If $f = f(s,t)$ is a function of two variables s and t, and we compute a definite integral **with respect to** t,

$$\int_a^b f(s,t)\,dt$$

we are left with a function of s alone.

Example: The integral[1]

$$\int_0^4 (2st - s)\,dt = 2s\,\frac{t^2}{2} - st\,\Big|_0^4 = 2s\,\frac{4^2}{2} - s\cdot 4 - \left(2s\cdot\frac{0^2}{2} - s\cdot 0\right)$$

$$= 16s - 4s = 12s$$

is a function of the variable s.

[1] The variable s is treated like a constant when integrating with respect to t—and visa versa.

Integral Transform

An **integral transform** is a mapping that assigns to a function $f(t)$ another function $F(s)$ via an integral of the form

$$\int_a^b K(s,t)f(t)\,dt.$$

- The function K is called the **kernel** of the transformation.
- The limits a and b may be finite or infinite.
- The integral may be improper so that convergence/divergence must be considered.
- This transform is **linear** in the sense that

$$\int_a^b K(s,t)(\alpha f(t)+\beta g(t))\,dt = \alpha\int_a^b K(s,t)f(t)\,dt + \beta\int_a^b K(s,t)g(t)\,dt.$$

The Laplace Transform

Definition: Let $f(t)$ be defined on $[0,\infty)$. The Laplace transform of f is denoted and defined by

$$\mathscr{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = F(s).$$

The domain of the transformation $F(s)$ is the set of all s such that the integral is convergent.

Note: The kernel for the Laplace transform is $K(s,t) = e^{-st}$.