March 31 Math 2335 sec 51 Spring 2016

Section 5.3: Gaussian Quadrature

Here we are going to approximate the integral $I(f)$ by the new *rule* called **Gaussian Quadrature**. The integration formula will be given by

$$I_n(f) = \sum_{j=1}^{n} w_j f(x_j)$$

where the numbers $\{w_1, \ldots, w_n\}$ are called the **weights** and $\{x_1, \ldots, x_n\}$ are called the **nodes.**

Main Idea: The weights and nodes are chosen so that $I_n(p) = I(p)$ exactly, for $p(x)$ any polynomial of degree as high as possible.

Gaussian Quadrature: $n = 1$ Case

When $n = 1$, the formula becomes

$$I_1(f) = \sum_{j=1}^{1} w_j f(x_j) = w_1 f(x_1).$$

There is one weight w_1 and one node x_1.

We determined that $w_1 = 2$ and $x_1 = 0$ giving the I_1 rule

$$\int_{-1}^{1} f(x)\,dx \approx I_1(f) = 2f(0).$$

Gaussian Quadrature: $n = 2$ Case

When $n = 2$, the formula becomes

$$I_2(f) = \sum_{j=1}^{2} w_j f(x_j) = w_1 f(x_1) + w_2 f(x_2).$$

There are two weights $\{w_1, w_2\}$ and two nodes $\{x_1, x_2\}$.

We determined that $w_1 = w_2 = 1$ with $x_1 = -\frac{1}{\sqrt{3}}$ and $x_2 = \frac{1}{\sqrt{3}}$. This gives the rule

$$\int_{-1}^{1} f(x)\,dx \approx I_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right).$$

Gaussian Quadrature: $I_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$

Use $I_2(f)$ to approximate $\int_{-1}^{1} \frac{dx}{1+x^2}$. Compare the result to the true value $\frac{\pi}{2}$.

$$f(x) = \frac{1}{1+x^2} \quad \text{(the integrand)}$$

$$I_2(f) = f\left(\frac{-1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{1+\left(\frac{-1}{\sqrt{3}}\right)^2} + \frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2} = \frac{1}{1+\frac{1}{3}} + \frac{1}{1+\frac{1}{3}} = \frac{3}{4} + \frac{3}{4}$$

$$= \frac{6}{4} = \frac{3}{2}$$

Gaussian Quadrature for $n > 2$

We would insist that the formula $I_n(p)$ is exact for $p(x) = 1, x, x^2, \ldots, x^{2n-1}$.[1] We'll get a nonlinear system of $2n$ equations

$$\begin{aligned}
w_1 + w_2 + \cdots + w_n &= 2\\
w_1x_1 + w_2x_2 + \cdots + w_nx_n &= 0\\
w_1x_1^2 + w_2x_2^2 + \cdots + w_nx_n^2 &= \frac{2}{3}\\
\vdots \quad &\quad \vdots\\
w_1x_1^{2n-2} + w_2x_2^{2n-2} + \cdots + w_nx_n^2 2n-2 &= \frac{2}{2n-1}\\
w_1x_1^{2n-1} + w_2x_2^{2n-1} + \cdots + w_nx_n^{2n-1} &= 0
\end{aligned}$$

[1] This is $2n$ conditions for the $2n$ unknowns $\{w_1, \ldots, w_n\}$ and $\{x_1, \ldots, x_n\}$.

Gaussian Quadrature for $n > 2$

Fortunately, solutions for various n values are known. Table 5.7 (pg. 223) in Atkinson and Han shows the weights and nodes for $n = 2, 3, \ldots, 8$.

For example, the weights and nodes for $I_3(f)$ are

$$w_1 = \frac{5}{9}, \quad w_2 = \frac{8}{9} \quad \text{and} \quad w_3 = \frac{5}{9}$$

$$x_1 = -\sqrt{\frac{3}{5}}, \quad x_2 = 0, \quad \text{and} \quad x_3 = \sqrt{\frac{3}{5}}.$$

Read the table carefully in the text. The weights for corresponding nodes are aligned in the table.

Example for $n = 3$

Use $I_3(f)$ to approximate $\int_{-1}^{1} \frac{dx}{1+x^2}$. Compare the result to the true value $\frac{\pi}{2}$.

$x_1 = -\sqrt{3/5} \quad w_1 = \frac{5}{9} \qquad x_2 = 0 \quad w_2 = \frac{8}{9} \qquad x_3 = \sqrt{\frac{3}{5}} \quad w_3 = \frac{5}{9}$

$$I_3(f) = \frac{5}{9} f\left(-\sqrt{3/5}\right) + \frac{8}{9} f(0) + \frac{5}{9} f\left(\sqrt{3/5}\right)$$

$$f(x) = \frac{1}{1+x^2}$$

$$f\left(\pm\sqrt{\frac{3}{5}}\right) = \frac{1}{1+\left(\pm\sqrt{3/5}\right)^2} = \frac{1}{1+3/5} = \frac{5}{8}$$

$$f(0) = \frac{1}{1+0^2} = 1$$

$$\int_{-1}^{1} \frac{dx}{1+x^2}\,dx \approx I_3(f) = \frac{5}{9}\cdot\frac{5}{8} + \frac{8}{9}\cdot 1 + \frac{5}{9}\cdot\frac{5}{8}$$

$$= \frac{25}{72} + \frac{64}{72} + \frac{25}{72} = \frac{114}{72}$$

$$= \frac{57}{36} = 1.58\overline{33}$$

$$\frac{\pi}{2} \doteq 1.570796$$

Other Intervals

Suppose we wish to evaluate

$$\int_a^b f(x)\,dx.$$

We'd like a change of variables $x \to t$ so that $-1 \le t \le 1$ when $a \le x \le b$. We already know that

$$t = \frac{2}{b-a}(x-a) - 1, \quad \text{i.e.} \quad x = \frac{b-a}{2}(t+1) + a$$

does the trick. This gives $dx = \frac{b-a}{2}dt$ so that

$$\int_a^b f(x)\,dx = \frac{b-a}{2}\int_{-1}^1 g(t)\,dt$$

where

$$g(t) = f\left(\frac{b-a}{2}(t+1) + a\right).$$

Example

Use $I_2(f)$ to approximate

$$\int_0^1 \sqrt{x}\,dx \qquad a=0 \quad \text{and} \quad b=1$$

$$x = \frac{b-a}{2}(t+1)+a = \frac{1-0}{2}(t+1)+0 = \frac{1}{2}(t+1)$$

$$f(x)=\sqrt{x} \quad \Rightarrow \quad g(t)=\sqrt{\frac{1}{2}(t+1)}$$

$$\int_0^1 \sqrt{x}\,dx = \frac{1-0}{2}\int_{-1}^{1}\sqrt{\frac{1}{2}(t+1)}\,dt = \frac{1}{2}\int_{-1}^{1}\sqrt{\frac{1}{2}(t+1)}\,dt$$

$$\approx I_2(g) = g\left(\frac{-1}{\sqrt{3}}\right) + g\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{1}{2}\sqrt{\frac{1}{2}\left(\frac{-1}{\sqrt{3}}+1\right)} + \frac{1}{2}\sqrt{\frac{1}{2}\left(\frac{1}{\sqrt{3}}+1\right)}$$

$$\doteq 0.67389$$

Comparison of Methods

$$\int_0^1 \sqrt{x}\, dx = \frac{2}{3} = 0.66667$$

$$I_2(f) = f\left(-\frac{1}{\sqrt{3}}\right) + f\left(\frac{1}{\sqrt{3}}\right) = 0.67389, \quad f(t) = \frac{1}{2}\sqrt{\frac{t+1}{2}}$$

$$T_4(f) = \frac{1}{8}\left[\sqrt{0} + 2\sqrt{\frac{1}{4}} + 2\sqrt{\frac{1}{2}} + 2\sqrt{\frac{3}{4}} + \sqrt{1}\right] = 0.64328$$

$$S_4(f) = \frac{1}{12}\left[\sqrt{0} + 4\sqrt{\frac{1}{4}} + 2\sqrt{\frac{1}{2}} + 4\sqrt{\frac{3}{4}} + \sqrt{1}\right] = 0.65653$$

The errors are

$$I - I_2 = -0.00722, \quad I - T_4 = 0.02338, \quad \text{and} \quad I - S_4 = 0.01014$$