March 3 Math 2306 sec 58 Spring 2016

Section 8: Homogeneous Equations with Constant Coefficients

We consider a linear, homogeneous equation with constant coefficients

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = 0.$$

We seek solutions of the form $y = e^{mx}$ for constant m, and obtain the characteristic (a.k.a. auxiliary) equation

$$a_n m^n + a_{n-1} m^{n-1} + \cdots + a_1 m + a_0 = 0$$

Higer Order Linear Constant Coefficient ODEs

- The same approach applies. For an n^{th} order equation, we obtain an n^{th} degree polynomial.
- Complex roots must appear in conjugate pairs (due to real coefficients) giving a pair of solutions $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$.
- If a root m is repeated k times, we get k linearly independent solutions

  $$e^{mx}, \quad xe^{mx}, \quad x^2e^{mx}, \quad \ldots, \quad x^{k-1}e^{mx}$$

  or in conjugate pairs cases $2k$ solutions

  $$e^{\alpha x}\cos(\beta x),\ e^{\alpha x}\sin(\beta x), \quad xe^{\alpha x}\cos(\beta x),\ xe^{\alpha x}\sin(\beta x), \ldots,$$
  $$x^{k-1}e^{\alpha x}\cos(\beta x),\ x^{k-1}e^{\alpha x}\sin(\beta x)$$

- It may require a computer algebra system to find the roots for a high degree polynomial.

Example

3rd order so we need y_1, y_2, y_3

Solve the ODE

$$y''' - 3y'' + 3y' - y = 0$$

Characteristic eqn.

$$m^3 - 3m^2 + 3m - 1 = 0$$

$$(m-1)^3 = 0 \Rightarrow m = 1 \text{ repeated root}$$

triple root

$$y_1 = e^x, \quad y_2 = xe^x, \quad y_3 = x^2 e^x$$

The general solution is

$$y = c_1 y_1 + c_2 y_2 + c_3 y_3$$

$$y = c_1 e^x + c_2 x e^x + c_3 x^2 e^x$$

Example

4th order, we need

y_1, y_2, y_3, y_4

Solve the ODE

$$\frac{d^4y}{dx^4}+2\frac{d^2y}{dx^2}+y=0$$

Characteristic eqn:

$$m^4+2m^2+1=0$$

$$(m^2+1)^2=0$$

$$(m-i)^2(m+i)^2=0$$

$$m=\pm i$$

the conjugate pair are double roots

side note

$m^2+1=0$

$m^2=-1$

$m=\pm i$

$\alpha \pm i\beta$ here $\alpha = 0$ and $\beta = 1$

$$y_1 = e^{0x}\cos(1x) = \cos x$$

$$y_2 = e^{0x}\sin(1x) = \sin x$$

$$y_3 = xe^{0x}\cos(1x) = x\cos x$$

$$y_4 = xe^{0x}\sin(1x) = x\sin x$$

The general solution is

$$\boxed{y = c_1\cos x + c_2\sin x + c_3 x\cos x + c_4 x\sin x.}$$

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Motivating Example

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We could ask what sort of function y might solve the ODE. Since $g(x)$ is a line, we might guess that y_p is a line.

Suppose $y_p = Ax + B$, A, B - constants

Substitute into the ODE

$y_p = Ax + B$

$y_p' = A$

$y_p'' = 0$

We require

$$y_p'' - 4y_p' + 4y_p = 8x + 1$$

$$0 - 4(A) + 4(Ax + B) = 8x + 1$$

$$4Ax + (-4A + 4B) = 8x + 1$$

This holds provided

$$4A = 8 \quad \text{and}$$

$$-4A + 4B = 1$$

i.e. $A = 2$ and $4B = 1 + 4A = 1 + 4 \cdot 2 = 9 \implies B = \frac{9}{4}$

So the solution

$$y_p = 2x + \frac{9}{4}$$

is a particular solution of the ODE.

The Method: Assume y_p has the same **form** as $g(x)$

$$y'' - 4y' + 4y = 6e^{3x}$$

Let's suppose that $y_p = Ae^{3x}$ for constant A.

$y_p = Ae^{3x}$

$y_p' = 3Ae^{3x}$

$y_p'' = 9Ae^{3x}$

we require

$$y_p'' - 4y_p' + 4y_p = 6e^{3x}$$

$$9Ae^{3x} - 4(3Ae^{3x}) + 4Ae^{3x} = 6e^{3x}$$

$$(9-12+4)Ae^{3x} = 6e^{3x}$$

$$Ae^{3x} = 6e^{3x}$$

This is true if $A = 6$.

A particular solution is

$$y_p = 6e^{3x}.$$

(That $g(x) = 6e^{3x}$ as well is a coincidence.)

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

a constant times x^2

more generally it's a quadratic.

Since $g(x)$ is a constant times x^2, let's guess that

$y_p = Ax^2$.

we require

$$y_p'' - 4y_p' + 4y_p = 16x^2$$

$$y_p = Ax^2$$

$$y_p' = 2Ax$$

$$y_p'' = 2A$$

$$2A - 4(2Ax) + 4(Ax^2) = 16x^2$$

$$4Ax^2 - 8Ax + 2A = 16x^2 + 0x + 0$$

This requires $4A=16$, $-8A=0$, $2A=0$

which requires $A=4$ AND $A=0$.

Since $g(x)$ is a quadratic, let's try again with $y_p = Ax^2 + Bx + C$

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$

$y_p'' = 2A$

Now

$y_p'' - 4y_p' + 4y_p = 16x^2$

$2A - 4(2Ax+B) + 4(Ax^2+Bx+C) = 16x^2$

$$4Ax^2 + (-8A + 4B)x + (2A - 4B + 4C) = 16x^2 + 0x + 0$$

The requires

$$4A = 16 \Rightarrow A = 4$$

$$-8A + 4B = 0 \Rightarrow 4B = 8A \Rightarrow B = 2A = 2\cdot 4 = 8$$

$$2A - 4B + 4C = 0$$

$$4C = 4B - 2A = 4\cdot 8 - 2\cdot 4 = 24$$

$$C = 6$$

So

$$\boxed{y_p = 4x^2 + 8x + 6}$$

General Form: sines and cosines

Constant times sin(2x)

$$y'' - y' = 20\sin(2x)$$

Let's guess that $y_p = A\sin(2x)$

$y_p = A\sin(2x)$

$y_p' = 2A\cos(2x)$

$y_p'' = -4A\sin(2x)$

we need

$y_p'' - y_p' = 20\sin(2x)$

$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x)$

This requires $-4A = 20$ and $-2A = 0$

A can't be -5 AND 0.

To correct this, we should assume

that $y_p = A\sin(2x) + B\cos(2x)$.