March 3 Math 2306 sec 59 Spring 2016

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Some Motivating Examples

Find a particular solution of the ODE

$$y'' - 4y' + 4y = 8x + 1$$

We made a guess that $y_p = Ax + B$, a line, since the right hand side is a line. We found that this worked with $A = 2$ and $B = 9/4$ giving the particular solution $y_p = 2x + 9/4$.

$$y'' - 4y' + 4y = 6e^{3x}$$

We made a guess that $y_p = Ae^{3x}$ which matches the basic form of the right hand side. We found that this worked with $A = 6$ giving the particular solution $y_p = 6e^{3x}$.

Make the form general

$$y'' - 4y' + 4y = 16x^2$$

With this one, we tried setting $y_p = Ax^2$, but we couldn't solve the equation for any value of A.

We corrected it to the form $y_p = Ax^2 + Bx + C$ and found that this does solve the equation if $A = 4$, $B = 8$ and $C = 6$. So we found the particular solution $y_p = 4x^2 + 8x + 6$.

General Form: sines and cosines

$$y'' - y' = 20\sin(2x)$$

← a constant times the $\sin(2x)$

Let's suppose $y_p = A\sin(2x)$.

$y_p = A\sin(2x)$

$y_p' = 2A\cos(2x)$

$y_p'' = -4A\sin(2x)$

we need

$y_p'' - y_p' = 20\sin(2x)$

$-4A\sin(2x) - 2A\cos(2x) = 20\sin(2x)$

Matching coefficients, we need

$-4A = 20$ <u>and</u> $-2A = 0$

We may need to match coefficients for $\cos(2x)$.

So let's guess $y_p = A\sin(2x) + B\cos(2x)$.

$$y_p' = 2A\cos(2x) - 2B\sin(2x)$$

$$y_p'' = -4A\sin(2x) - 4B\cos(2x)$$

we require $y_p'' - y_p' = 20\sin(2x)$

$$-4A\sin(2x) - 4B\cos(2x) - \big(2A\cos(2x) - 2B\sin(2x)\big) = 20\sin(2x)$$

$$(\underline{-4A+2B})\sin(2x) + (\underline{\underline{-2A-4B}})\cos(2x) = \underline{20}\sin(2x) + \underline{\underline{0}}\cdot\cos(2x)$$

$$-4A + 2B = 20$$
$$-2A - 4B = 0$$

$$\Rightarrow \quad \begin{aligned} -8A + 4B &= 40 \\ -2A - 4B &= 0 \\ \hline -10A &= 40 \end{aligned} \quad \text{add} \quad \Rightarrow A = -4$$

$$-4B = 2A = 2(-4) = -8 \Rightarrow B = 2$$

$$\boxed{\begin{array}{l}\text{So we have a particular solution} \\ \qquad y_p = -4\sin(2x) + 2\cos(2x)\end{array}}$$

Examples of Forms of y_p based on g (Trial Guesses)

(a) $g(x) = 1$ (or really any nonzero constant)

Guess $y_p = A$

(b) $g(x) = x - 7$ 1st degree poly. $y_p = Ax + B$

(c) $g(x) = 5x$ 1st degree poly. $y_p = Ax + B$

(d) $g(x) = 3x^3 - 5$ 3rd degree poly

$y_p = Ax^3 + Bx^2 + Cx + D$

More Trial Guesses

(e) $g(x) = xe^{3x}$

1st degree Poly. times e^{3x}

$y_p = (Ax+B)e^{3x} = Axe^{3x} + Be^{3x}$

(f) $g(x) = \cos(7x)$

Sum of sines / cosines of $7x$

$y_p = A\cos(7x) + B\sin(7x)$

(g) $g(x) = \sin(2x) - \cos(4x)$

Sine / Cosines of $2x$ and sines / cosines of $4x$

$y_p = A\sin(2x) + B\cos(2x) + C\cos(4x) + D\sin(4x)$

(h) $g(x) = x^2\sin(3x)$

2nd degree poly times sines / cosines $3x$

$y_p = (Ax^2 + Bx + C)\sin(3x) + (Dx^2 + Ex + F)\cos(3x)$

Still More Trial Guesses

(i) $g(x) = e^x \cos(2x)$

$y_p = Ae^x \cos(2x) + Be^x \sin(2x)$

(j) $g(x) = x^3 e^{8x}$

$y_p = (Ax^3 + Bx^2 + Cx + D)\, e^{8x}$

(k) $g(x) = xe^{-x} \sin(\pi x)$

$y_p = (Ax+B)e^{-x} \sin(\pi x) + (Cx+D)e^{-x} \cos(\pi x)$.

The Superposition Principle

$$y'' - y' = 20\sin(2x) + 4e^{-5x}$$

we can consider $y'' - y' = g_1(x) + g_2(x)$

where $g_1(x) = 20\sin(2x)$ and $g_2(x) = 4e^{-5x}$

For $y'' - y' = 20\sin(2x)$, we know $y_{p_1} = -4\sin 2x + 2\cos 2x$

For $y'' - y' = 4e^{-5x}$, guess $y_{p_2} = Ae^{-5x}$

$y_{p_2}' = -5Ae^{-5x}$, $y_{p_2}'' = 25Ae^{-5x}$

$$y_{p_2}'' - y_{p_2}' = 4e^{-5x}$$

$$25Ae^{-5x} - (-5A)e^{-5x} = 4e^{-5x}$$

$$30Ae^{-5x} = 4e^{-5x} \implies A = \frac{4}{30} = \frac{2}{15}$$

So $y_{p_2} = \frac{2}{15}e^{-5x}$.

Using superposition.

$$y_p = y_{p_1} + y_{p_2} \implies \underline{\underline{y_p = -4\sin(2x) + 2\cos(2x) + \frac{2}{15}e^{-5x}}}.$$