March 3 Math 2335 sec 51 Spring 2016

Section 4.2: Error in Polynomial Interpolation

We consider a polynomial interpolation P_n for a set of data $\{(x_i, y_i),\ i = 0, \ldots n \,|\, y_i = f(x_i)\}$.

Theorem: For $n \geq 0$, suppose f has $n+1$ continuous derivatives on $[a, b]$ and let $x_0, \ldots, x_n$ be distinct nodes in $[a, b]$. Then

$$f(x) - P_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(c_x)}{(n+1)!}$$

where c_x is some number between the smallest and largest values of $x_0, \ldots, x_n$ and x.

Error $f(x) - P_n(x)$

The error can be restated as

$$\text{Err}(P_n(x)) = \Psi_n(x)\,\frac{f^{(n+1)}(c_x)}{(n+1)!}$$

where Ψ_n is the $n+1$ degree monic polynomial

$$\Psi_n(x) = (x - x_0)\cdots(x - x_n) = x^{n+1} + \text{terms with smaller powers}$$

The coefficients of those *smaller powers* depend on $x_0, \ldots, x_n$.

The error depends on the y's due to $f^{(n+1)}(c_x)$, and on the x's due to $\Psi_n(x)$.

Error in $\Psi_n(x)$: Equally Spaced Nodes

We found the general results for $n = 1$ and $n = 2$ if the points are all h units apart:

- For $|x_1 - x_0| = h$, the maximum value of $|\Psi_1(x)|$ is $M = \frac{h^2}{4}$ for x between x_0 and x_1.

- For $h = |x_2 - x_1| = |x_1 - x_0|$, the maximum value of $|\Psi_2(x)|$ is $M = \frac{2h^3}{3\sqrt{3}}$ for x between the largest and smallest of x_0, x_1 and x_2.

A slightly more complicated analysis can be used to show that for equally spaced nodes (in order) $\{x_0, \ldots, x_3\}$ each h units apart,

$$|\Psi_3(x)| \le h^4 \quad \text{for} \quad x_0 \le x \le x_3.$$

Example

Use the result $|\Psi_3(x)| \leq h^4$ for $x_0 \leq x \leq x_3$. for equally spaced nodes to bound the error.

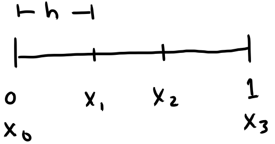
Suppose that $P_3(x)$ is used to approximate $f(x) = e^x$ over the interval $[0, 1]$ using equally spaced points with $x_0 = 0$, $x_3 = 1$. Find a bound on the error

$$|e^x - P_3(x)|$$

$$|e^x - P_3(x)| = \left| \Psi_3(x) \frac{f^{(4)}(c_x)}{4!} \right|$$

For $f(x) = e^x$, $f^{(4)}(x) = e^x$ so $f^{(4)}(c_x) = e^{c_x}$

For $0 \le c_x \le 1$ $\quad e^0 \le e^{c_x} \le e^1 = e$



so $h = \frac{1}{3}$

$$|\psi_3(x)| \le h^4 = \left(\frac{1}{3}\right)^4 = \frac{1}{81}$$

so for x in $[0,1]$

$$\left|e^x - P_3(x)\right| = \left|\psi_3(x)\,\frac{e^{c_x}}{4!}\right| \le \frac{1}{81}\cdot\frac{e}{24} \doteq 0.0013$$

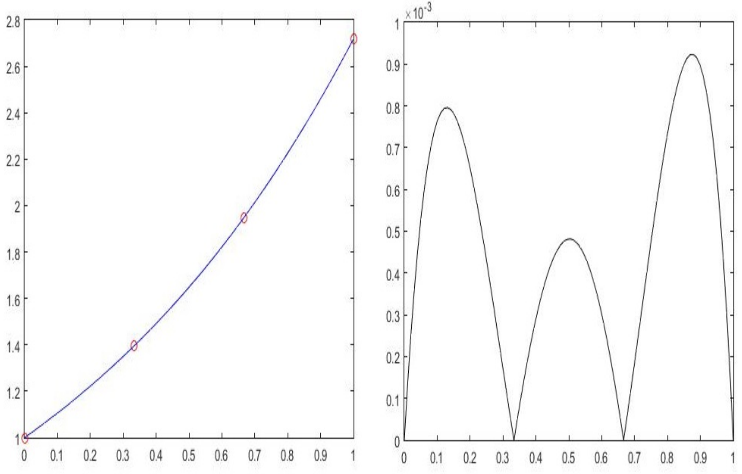


Figure: Left: Plot of P_3 for $f(x) = e^x$ with nodes (in red) at $\{0, 1/3, 2/3, 1\}$. Right: Plot of the error $|e^x - P_3(x)|$. The function P_3 was generated numerically using the Matlab program interpNDD.m available in D2L.

Behavior of $\Psi(x)$

If we can choose our nodes, an obvious choice is to make them equally spaced. But the question arises:

Question: Are equally spaced nodes the best choice for minimizing error?

(Here we're going to discuss section 4.2.2, then move to section 4.5, and later come back to section 4.3.)

Motivating Example

Suppose we wish to use $P_4(x)$ to interpolate a function $f(x)$ on the interval $[-1, 1]$. We know that the error

$$|f(x) - P_4(x)| = \left|(x - x_0)\cdots(x - x_4)\,\frac{f^{(5)}(c_x)}{5!}\right| \leq ML$$

where

$$L = \max\left|\frac{f^{(5)}(c_x)}{5!}\right| \quad \text{and} \quad M = \max|(x - x_0)\cdots(x - x_4)|$$

Motivating Example

Let's see what kind of control we may have over M. We can consider two examples of the function $(x - x_0)\cdots(x - x_4)$ over the interval $[-1, 1]$.

Equally Spaced Points: $\Psi_{4,1}(x) = (x+1)(x+\frac{1}{2})x(x-\frac{1}{2})(x-1)$

Not Equally Spaced:
$\Psi_{4,2}(x) = (x - x_0)(x - x_1)(x - x_2)(x - x_3)(x - x_4)$ where

$$x_0 = \cos\left(\frac{\pi}{10}\right) \approx 0.9511, \quad x_1 = \cos\left(\frac{3\pi}{10}\right) \approx 0.5878,$$

$$x_2 = \cos\left(\frac{5\pi}{10}\right) = 0, \quad x_3 = \cos\left(\frac{7\pi}{10}\right) \approx -0.5878,$$

$$\text{and} \quad x_4 = \cos\left(\frac{9\pi}{10}\right) \approx -0.9511$$

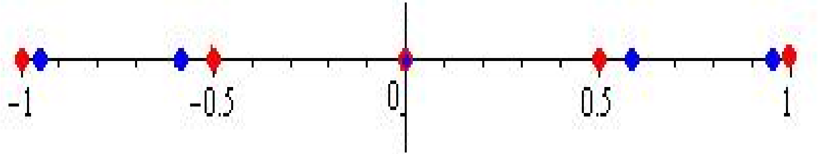


Figure: Two choices of nodes for P_4 on $[-1, 1]$. Red dots are equally spaced nodes, and blue dots are an alternative choice (Chebyshev nodes).

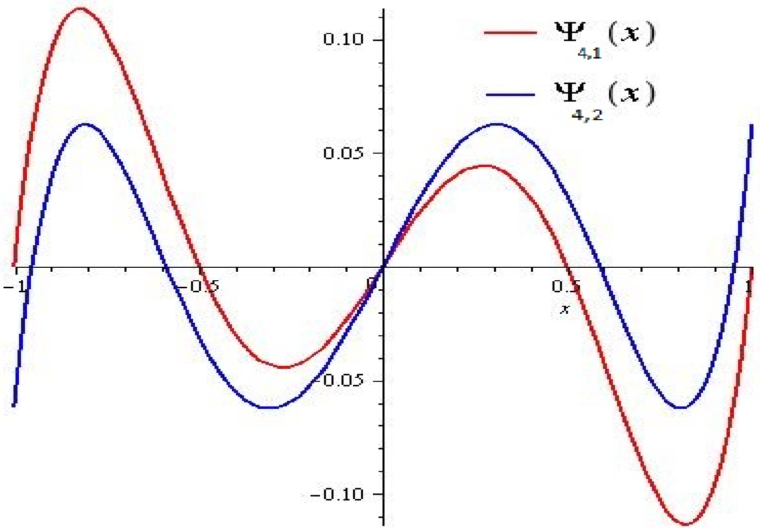


Figure: Plot of $\Psi_{4,1}(x)$ and $\Psi_{4,2}(x)$ shows that $\Psi_{4,2}$ has a smaller maximum value.

Motivating Example Continued...

The maximum value of $\Psi_{4,1}(x)$ is ≈ 0.1135. The maximum value of $\Psi_{4,2}(x)$ is 0.0625.

The error when using equally spaced nodes is 1.8 times as great as the error when using the alternative choice of nodes!

Error for Equally Spaced Nodes

When equally spaced nodes are used, the behavior at the ends (near a and b) can be quite dramatic. The error for x in the middle may be small, while the error for x near the ends is much larger.

If $f^{(n+1)}(x)$ is ill behaved, it's possible that taking n larger results in more error rather than less!

A special case of this is the function

$$f(x) = \frac{1}{1+x^2} \quad \text{for} \quad -5 \le x \le 5$$

(See the next two slides.)

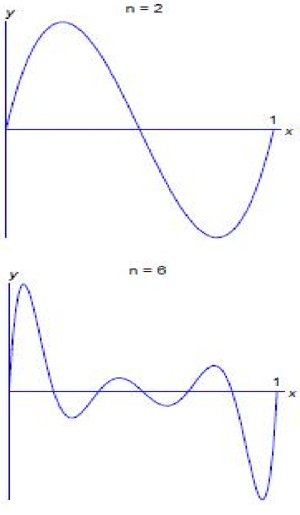


Figure: Plots of $(x - x_0) \cdots (x - x_n)$ for equally spaced nodes on $[0, 1]$ for $n = 2, 4, 6$ and 8. Note that the local extrema seem to get pushed toward the ends of the interval as n increases.

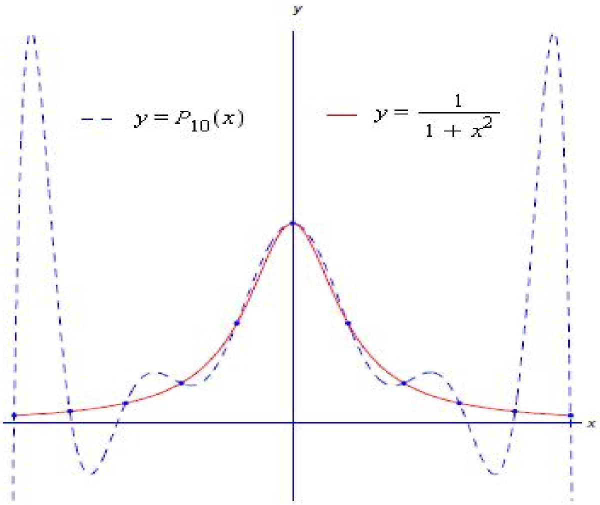


Figure: Plot of $y = \frac{1}{1+x^2}$ (red) together with degree 10 polynomial interpolation $P_{10}(x)$ (blue dash) obtained using equally spaced nodes on $[-5, 5]$.

Alternatives to Equally Spaced Nodes

Recall that for the example using P_4 we considered the seemingly strange nodes

$$x_0 = \cos\left(\frac{\pi}{10}\right) \approx 0.9511, \quad x_1 = \cos\left(\frac{3\pi}{10}\right) \approx 0.5878,$$

$$x_2 = \cos\left(\frac{5\pi}{10}\right) = 0, \quad x_3 = \cos\left(\frac{7\pi}{10}\right) \approx -0.5878,$$

$$\text{and} \quad x_4 = \cos\left(\frac{9\pi}{10}\right) \approx -0.9511$$

It turns out that there is a motivation for using these even though they appear more complicated than just taking equally spaced ones.

Alternatives to Equally Spaced Nodes

When can one **choose** nodes:

- when picking a partition for numerical integration (see chapter 5 in Atkinson & Han).
- when choosing a grid for computer generated graphics
- when interpolating a function when nodes are not pre-specified.

Of course, when presented with raw data, one may not have the option of picking one's nodes.

Section 4.5 (& 4.6): Chebyshev Polynomials

Definition: For an integer $n \geq 0$ define the function

$$T_n(x) = \cos\left(n\ \cos^{-1}(x)\right), \quad -1 \leq x \leq 1.$$

It can be shown that T_n is a polynomial of degree n. It's called the

Chebyshev Polynomial of degree n.

Chebyshev Polynomials

Determine the polynomials $T_0(x)$, $T_1(x)$, and $T_2(x)$ in the form of ordinary polynomials.

$$T_n(x) = \cos(n \cos^{-1} x) \quad \text{for} \quad -1 \le x \le 1$$

$$T_0(x) = \cos(0 \cdot \cos^{-1} x) = \cos(0) = 1 \qquad T_0(x) = 1$$

$$T_1(x) = \cos(1 \cdot \cos^{-1} x) = \cos(\cos^{-1} x) = x \qquad T_1(x) = x$$

$$T_2(x) = \cos(2 \cos^{-1} x)$$

Recall $\cos(2\theta) = 2\cos^2\theta - 1$

So $T_2(x) = 2\cos^2(\cos^{-1}x) - 1$

$$= 2\left[\cos(\cos^{-1}x)\right]^2 - 1$$

$$= 2[x]^2 - 1$$

i.e, $T_2(x) = 2x^2 - 1$

Recursion Relation

$T_0(x) = 1$ and $T_1(x) = x$. It can be shown that for $n \geq 1$

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x).$$

Compute $T_2(x)$ and $T_3(x)$ using this relation.

$$T_2(x) = 2x\,T_1(x) - T_0(x) = 2x(x) - 1 = 2x^2 - 1$$

$$T_2(x) = 2x^2 - 1$$

$$T_3(x) = 2x\,T_2(x) - T_1(x) = 2x(2x^2 - 1) - x = 4x^3 - 2x - x$$

$$T_3(x) = 4x^3 - 3x$$

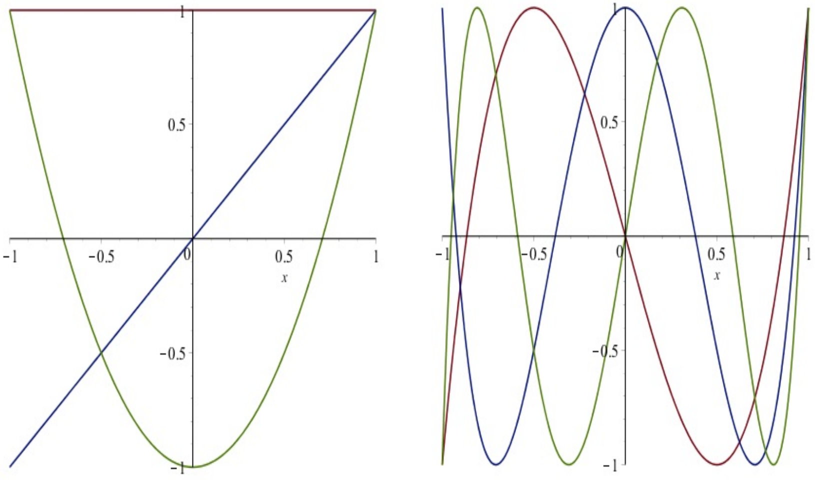


Figure: Plot of the first six Chebyshev Polynomials (of the first kind). T_0, T_1, and T_2 are shown on the left, and T_3, T_4, and T_5 are shown on the right.

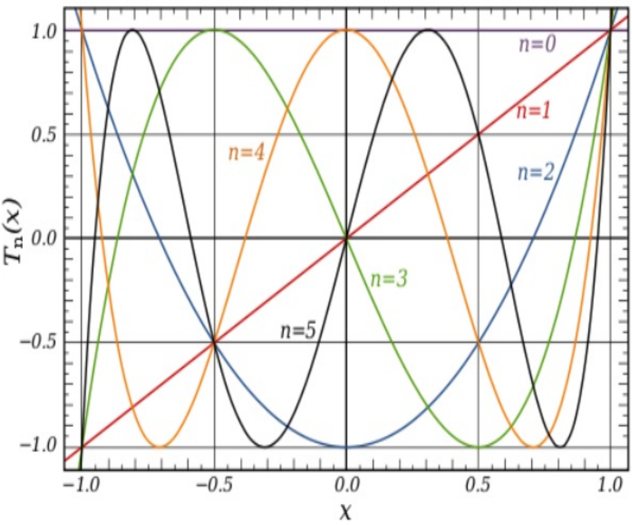


Figure: Plot of all of the first six Chebyshev polynomials (of the first kind).

Some Properties of Chebyshev Polynomials

- T_n is an even function if n is even and an odd function if n is odd.
- $T_n(1) = 1$ and $T_n(-1) = (-1)^n$ for every n
- They have an orthogonality relation

$$\int_{-1}^{1} \frac{T_n(x)T_m(x)}{\sqrt{1-x^2}} dx = 0 \quad n \neq m.$$

- **And the main property we're interested in**

$$|T_n(x)| \leq 1 \quad \text{for all} \quad -1 \leq x \leq 1$$

Minimum Size Property

We can note that

$$T_n(x) = 2^{n-1}x^n + \text{terms with lower powers.}$$

We define the **modified Chebyshev polynomials** by

$$\tilde{T}_n(x) = \frac{1}{2^{n-1}} T_n(x).$$

Remark: The modified Chebyshev polynomials are monic polynomials. That is

$$\tilde{T}_n(x) = x^n + \text{terms with lower powers.}$$

Minimum Size Property

Theorem: Let $n \geq 1$ be an integer. Of all monic polynomials on the interval $[-1, 1]$, the one with the smallest maximum value is the modified Chebyshev polynomial $\tilde{T}_n(x)$. Moreover

$$|\tilde{T}_n(x)| \leq \frac{1}{2^{n-1}} \quad \text{for all} \quad -1 \leq x \leq 1.$$

This result suggests that whenever possible, we choose the polynomial $\Psi_n(x)$ in our error theorem to be the modified Chebyshev polynomial $\tilde{T}_{n+1}(x)$.

Chebyshev Nodes

Since $\tilde{T}_{n+1}(x)$ is monic, it can be written as

$$\tilde{T}_{n+1}(x) = (x - r_0)(x - r_1)\cdots(x - r_n)$$

where $r_0, \ldots, r_n$ are the roots of $T_{n+1}(x)$.

We had the polynomial in our error formula

$$\Psi_n(x) = (x - x_0)(x - x_1)\cdots(x - x_n).$$

So to minimize the error—i.e. make $\Psi_n(x) = \tilde{T}_{n+1}(x)$—we would have to

choose the nodes x_j to be the roots r_j of the Chebyshev polynomial T_{n+1}.

Example: Chebyshev Nodes

Use the change of variables $x = \cos\theta$ to find the five roots of $T_5(x)$.

$$T_5(x) = \cos\left(5\ \cos^{-1}(x)\right)$$

For $x = \cos\theta$, $\theta = \cos^{-1}x$ for $0 \le \theta \le \pi$

So $T_5(x) = 0$ if $\cos(5\theta) = 0$

This requires $5\theta = \frac{\pi}{2} + j\pi$

$$\theta = \frac{\pi/2 + j\pi}{5} = \frac{\pi}{10} + \frac{j\pi}{5} = \frac{\pi + 2\pi j}{10}$$

We get all 5 roots letting $j = 0, 1, 2, 3, 4$

$$\theta_0 = \frac{\pi}{10}, \quad \theta_1 = \frac{\pi + 2\pi}{10} = \frac{3\pi}{10}, \quad \theta_2 = \frac{\pi + 4\pi}{10} = \frac{\pi}{2}$$

$$\theta_3 = \frac{\pi + 6\pi}{10} = \frac{7\pi}{10}, \quad \theta_4 = \frac{\pi + 8\pi}{10} = \frac{9\pi}{10}$$

The roots are (from $x = \cos\theta$)

$$x_0 = \cos\left(\frac{\pi}{10}\right), \quad x_1 = \cos\left(\frac{3\pi}{10}\right), \quad x_2 = \cos\left(\frac{\pi}{2}\right) = 0$$

$$x_3 = \cos\left(\frac{7\pi}{10}\right) \quad \text{and} \quad x_4 = \cos\left(\frac{9\pi}{10}\right)$$

Chebyshev Nodes

Find a formula for the k roots of $T_k(x) = \cos\left(k \ \cos^{-1}(x)\right)$.

Again letting $x = \cos\theta$ i.e. $\theta = \cos^{-1}x$ for $0 \le \theta \le \pi$

$T_k(x) = 0$ if $\cos(k\theta) = 0$

$$k\theta = \frac{\pi}{2} + j\pi \Rightarrow \theta = \frac{\pi + 2j\pi}{2k}$$

We get k roots letting

$j = 0, 1, \ldots, k-1$

The roots are the x values where $x = \cos\theta$

$$x_j = \cos\left(\frac{\pi + 2j\pi}{2k}\right) \quad \text{for } j = 0, 1, \ldots, k-1$$

Chebyshev Nodes

To interpolate $f(x)$ on the interval $[-1, 1]$ by $P_n(x)$, the error is minimized by choosing the Chebyshev nodes (roots of $T_{n+1}(x)$)

$$x_j = \cos\left(\frac{(2j+1)\pi}{2(n+1)}\right), \quad j = 0, 1, \ldots, n.$$

The resulting error bound is

$$|f(x) - P_n(x)| \le \frac{L}{2^n}, \quad \text{where} \quad L = \max_{-1 \le x \le 1} \left| \frac{f^{(n+1)}(x)}{(n+1)!} \right|$$

Example

Let $f(x) = e^{2x}$ on $[-1, 1]$. Determine the Chebyshev nodes if $P_3(x)$ is being used to approximate $f(x)$, and determine the resulting error bound.

The nodes are the roots of $T_4(x)$

$$x_j = \cos\left(\frac{\pi + 2j\pi}{2\cdot 4}\right), \quad j = 0, 1, 2, 3$$

$$= \cos\left(\frac{\pi + 2j\pi}{8}\right)$$

$$x_0 = \cos\left(\frac{\pi}{8}\right) \doteq 0.9239 \qquad x_1 = \cos\left(\frac{3\pi}{8}\right) \doteq 0.3827$$

$$x_2 = \cos\left(\frac{5\pi}{8}\right) \doteq -0.3827 \qquad x_3 = \cos\left(\frac{7\pi}{8}\right) \doteq -0.9239$$

$$|f(x) - P_3(x)| \leq \frac{L}{2^3} \quad \text{where} \quad L = \max_{\text{on } [-1,1]} \left| \frac{f^{(4)}(x)}{4!} \right|$$

$$f(x) = e^{2x} \quad \text{so} \quad f^{(4)}(x) = 2^4 e^{2x}$$

$$\text{for} \quad -1 \leq x \leq 1 \qquad e^{-2} \leq e^{2x} \leq e^{2}$$

so

$$|f(x) - P_3(x)| \leq \frac{\frac{2^4 e^2}{4!}}{2^3} = \frac{2e^2}{24} = \frac{e^2}{12} \doteq 0.616$$