March 4 Math 3260 sec. 51 Spring 2020

Section 4.2: Null & Column Spaces, Linear Transformations

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted[1] by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

We can say that Nul A is the subset of $\mathbb{R}^n$ that gets mapped to the zero vector under the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

[1] Some authors will write Null(A) with two ells.

Theorem

For $m \times n$ matrix A, Nul A is a subspace of $\mathbb{R}^n$.

- Obviously, $A\mathbf{0} = \mathbf{0}$. So $\mathbf{0}$ is in Nul$A$.
- On the first exam, you showed that linear combinations of solutions to a homogenous equation are also solutions to that homogeneous equation.

So that establishes the necessary three properties for being a subspace. As the next example shows, it is always possible to express Nul$A$ as a span.

Example

For a given matrix, a spanning set for Nul$A$ gives an *explicit* description of this subspace. Find a spanning set for Nul A where

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 2 & 6 & -5 \end{bmatrix}.$$

We'll use an rref

$$A \rightarrow \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$$A\vec{x} = \vec{0} \quad \Rightarrow \quad \begin{aligned} x_1 &= -2x_3 + x_4 \\ x_2 &= -2x_3 + 2x_4 \\ & x_3, x_4 - \text{free} \end{aligned}$$

So solutions $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$

We can use the two vectors to set the explicit description

$$\mathrm{Nul}\, A = \mathrm{Span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Column Space

Definition: The **column space** of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\mathrm{Col} A = \mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Note that this corresponds to the set of solutions $\mathbf{b}$ of linear equations of the form $A\mathbf{x} = \mathbf{b}$! That is

$$\mathrm{Col} A = \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

This is true by virtue of Col A being the span of a set of vectors in $\mathbb{R}^m$.

Corollary: Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Example

Find a matrix A such that $W = \text{Col } A$ where

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

We'd like to take an arbitrary element of W and write it as $A\vec{x}$ for some matrix A.

$$\begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} = \begin{bmatrix} 6a \\ a \\ -7a \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix}$$

$$= a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix}$$

If we set $A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$, then

$W = \text{Col}\, A$.

Example

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

(a) If Col A is a subspace of $\mathbb{R}^k$, what is k?

$k=3$, the columns of A are in $\mathbb{R}^3$

(b) If Nul A is a subspace of $\mathbb{R}^k$, what is k?

$A\vec{x}$ $\quad k=4 \quad$ $A\vec{x}$ is defined for vectors in $\mathbb{R}^4$

$3\times 4 \quad 4\times 1$

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

(c) Is **u** in Nul A? Could **u** be in Col A?

① Is $A\vec{u} = \vec{0}$?

② $A\vec{u} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \vec{0}$

$\vec{u}$ is not in Nul A.

② Is $A\vec{x} = \vec{u}$ consistent?

$\underbrace{3\times 4 \quad 4\times 1}_{3\times 1}$

$\vec{u}$ is in $\mathbb{R}^4$, Col A is a subspace of $\mathbb{R}^3$, so

No, $\vec{u}$ is not in Col A.

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(c) Is $\mathbf{v}$ in Col A? ① Could $\mathbf{v}$ be in Nul A? ②

① Is $A\vec{x} = \vec{v}$ consistent? $[A \; \vec{v}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & 0 & -30/17 \\ 0 & 0 & 0 & 1 & 1/17 \end{bmatrix}$

not a pivot column

Yes, $A\vec{x} = \vec{v}$ is consistent, $\vec{v}$ is in Col A.

② Is $A\vec{v} = \vec{0}$? No, $A\vec{v}$ isn't defined since $\vec{v}$ is in $\mathbb{R}^3$. No, $\vec{v}$ is not in Nul A.

Linear Transformation

Definition: Let V and W be vector spaces. A linear transformation $T : V \longrightarrow W$ is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

(i) $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every $\mathbf{u}$ in V and scalar c.

Example: Differentiation

Recall that we defined the set $C^1(\mathbb{R})$ as the set of all real valued functions with domain $\mathbb{R}$ that are one-times continuously differentiable.

A function f is in $C^1(\mathbb{R})$ if

- $f'(x)$ exists, and
- $f'(x)$ is continuous on $(-\infty, \infty)$.

Let $C^0(\mathbb{R})$ denote the set of all real valued functions that are continuous on $\mathbb{R}$.

A function f is in $C^0(\mathbb{R})$ if $f(x)$ is continuous on $(-\infty, \infty)$.

Example: Differentiation[2]

Define the transformation D by

$$D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad D(f) = f'$$

Show that D is a linear transformation.

From Calc I $\frac{d}{dx}(f(x)+g(x)) = f'(x) + g'(x)$

$D(f+g) = (f+g)' = f' + g' = D(f) + D(g)$

Also recall $\frac{d}{dx}(c f(x)) = c f'(x)$, so

$D(cf) = (cf)' = c f' = c D(f)$

So D is a linear transformation.

[2] We can write $D(f(x)) = \frac{df}{dx}$.

Example

Characterize the subset[3] of $C^1(\mathbb{R})$ such that $D(f) = 0$.

If $f'(x) = 0$ for all x then

$f(x) = k$ for some constant k.

$D(f) = 0$ requires $f(x)$ is constant.

The subset is the set of all

constant functions.

[3] The zero vector in $C^0(\mathbb{R})$ is the function $z(x) = 0$ for all x.

Range and Kernel

Definition: The **range** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. (The set of all images of elements of V.)

Definition: The **kernel** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors $\mathbf{x}$ in V such that $T(\mathbf{x}) = \mathbf{0}$. (The analog of the null space of a matrix.)

Theorem: Given linear transformation $T : V \longrightarrow W$, the range of T is a subspace of W and the kernel of T is a subspace of V.

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(a) Express the equation that a function y must satisfy if y is in the kernel of T.

y is in the kernel if $T(y) = 0$

$$T(y) = \frac{dy}{dx} + \alpha y.$$

The equation is

$$\frac{dy}{dx} + \alpha y = 0$$

Example $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$

$$T(f) = \frac{df}{dx} + \alpha f(x) \quad \alpha \text{ a fixed constant.}$$

(b) Show that for any scalar c, $y = ce^{-\alpha x}$ is in the kernel of T.

We have to show that such y satisfies

$\frac{dy}{dx} + \alpha y = 0$, Set $y = c e^{-\alpha x}$,

then $\frac{dy}{dx} = c e^{-\alpha x}(-\alpha) = -\alpha c e^{-\alpha x}$

So $\frac{dy}{dx} + \alpha y = -\alpha c e^{-\alpha x} + \alpha(c e^{-\alpha x}) = 0$

$\therefore$ Yes, $y = ce^{-\alpha x}$ is in the kernel of T.