March 6 MATH 1112 sec. 52 Spring 2020

Definition:

For all real numbers x, the inverse tangent of x is denoted by

$$\tan^{-1}(x) \quad \text{or by} \quad \arctan(x)$$

and is defined by the relationship

$$y = \tan^{-1}(x) \iff x = \tan(y) \quad \text{where} \quad -\frac{\pi}{2} < y < \frac{\pi}{2}.$$

Quad I and IV

The Domain of the Inverse Tangent is $-\infty < x < \infty$.

The Range of the Inverse ~~Cosine~~ is $-\frac{\pi}{2} < y < \frac{\pi}{2}$ (Note the strict inequalities.).

Tangent

Conceptual Definition

We can think of the inverse tangent function in the following way:

$\tan^{-1}(x)$ is the *angle* between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ whose tangent is x.

quad I and $\underline{\text{IV}}$

Function/Inverse Function Relationship

For all real numbers x

$$\tan\left(\tan^{-1}(x)\right) = x$$

For every x in the interval $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$$\tan^{-1}(\tan(x)) = x$$

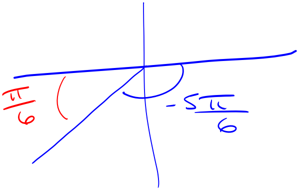
Remark 1: The expression $\tan^{-1}(x)$ is always well defined.

Remark 2: If $x > \frac{\pi}{2}$ or $x < -\frac{\pi}{2}$, the expression $\tan^{-1}(\tan(x))$ MAY BE defined, but IS NOT equal to x.

Examples

Evaluate each expression if possible. If undefined, state a reason.
Note $\tan^{-1}(\tan x)$ is only x if $-\frac{\pi}{2} < x < \frac{\pi}{2}$.

$$\tan^{-1}\left[\tan\left(-\frac{5\pi}{6}\right)\right]$$



$$= \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

$$\tan\left(\frac{-5\pi}{6}\right) = \pm\tan\left(\frac{\pi}{6}\right)$$

+ quad III

$$= \frac{1}{\sqrt{3}}$$

Combining Functions

Evaluate each expression exactly if possible. If it is undefined, provide a reason.

$$\tan\left[\arcsin\left(\frac{1}{2}\right)\right]$$

$$\tan\left(\frac{\pi}{6}\right)$$

$$= \frac{1}{\sqrt{3}}$$

$\arcsin\left(\frac{1}{2}\right) = \frac{\pi}{6}$

"angle in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ whose sine is $\frac{1}{2}$"

Combining Functions

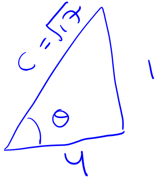
$$\sin\left[\tan^{-1}\left(\frac{1}{4}\right)\right]$$

$$= \frac{1}{\sqrt{17}}\left(\frac{\sqrt{17}}{\sqrt{17}}\right)$$

$$= \frac{\sqrt{17}}{17}$$

Call $\tan^{-1}\left(\frac{1}{4}\right) = \theta$

$$\tan\theta = \frac{1}{4} = \frac{\text{opp}}{\text{adj}}$$



$$4^2+1^2 = c^2$$

$$16+1 = c^2 \Rightarrow c = \sqrt{17}$$

$$\sin\theta = \frac{1}{\sqrt{17}}$$

Algebraic Expressions

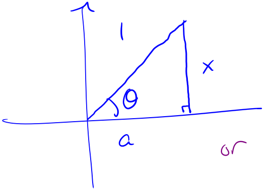
Write the expression purely algebraically (with no trigonometric functions).

$$\tan\left[\sin^{-1}(x)\right] = \frac{x}{\sqrt{1-x^2}}$$

Let $\theta = \sin^{-1} x$

$\sin\theta = x = \frac{x}{1}$

and $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$



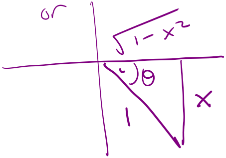
$\tan\theta = \frac{\text{opp}}{\text{adj}} = \frac{x}{\sqrt{1-x^2}}$

$a^2 + x^2 = 1^2$

$a^2 = 1^2 - x^2$

$a = \sqrt{1-x^2}$

or



Recap: Inverse Sine, Cosine, and Tangent

Function	Domain	Range
$\sin^{-1}(x)$	$[-1, 1]$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
$\cos^{-1}(x)$	$[-1, 1]$	$[0, \pi]$
$\tan^{-1}(x)$	$(-\infty, \infty)$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

Three Other Inverse Trigonometric Functions

There is disagreement about how to define the ranges of the inverse cotangent, cosecant, and secant functions!

The inverse Cotangent function is **typically** defined for all real numbers x by

$$y = \cot^{-1}(x) \quad \iff \quad x = \cot(y) \quad \text{for} \quad 0 < y < \pi.$$

There is less consensus regarding the inverse secant and cosecant functions.

A Work-Around

Use the fact that $\sec\theta = \frac{1}{\cos\theta}$ to find an expression for $\sec^{-1}(x)$.

Let $y = \sec^{-1}(x) \Rightarrow x = \sec y$

$$\Rightarrow \frac{1}{x} = \frac{1}{\sec y} = \cos y$$

$$\cos^{-1}\left(\frac{1}{x}\right) = \cos^{-1}(\cos y) = y$$

Use your result to compute (*an acceptable value for*) $\sec^{-1}(\sqrt{2})$

$$\sec^{-1}(\sqrt{2}) = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$$

A Work-Around

We can use the following compromise to compute inverse cotangent, secant, and cosecant values[1]:

For those values of x for which each side of the equation is defined

$$\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right),$$

$$\csc^{-1}(x) = \sin^{-1}\left(\frac{1}{x}\right),$$

$$\sec^{-1}(x) = \cos^{-1}\left(\frac{1}{x}\right).$$

[1] **Keep in mind that there is disagreement about the ranges!**

Question

True/False: Since $\cot\theta = \dfrac{1}{\tan\theta}$, it must be that $\cot^{-1} x = \dfrac{1}{\tan^{-1} x}$.

(a) True, and I'm confident.

(b) True, but I'm not confident.

False!

(c) False, and I'm confident.

(d) False, but I'm not confident.