

## Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where  $g$  comes from the restricted classes of functions

- ▶ polynomials,
- ▶ exponentials,
- ▶ sines and/or cosines,
- ▶ and products and sums of the above kinds of functions

Recall  $y = y_c + y_p$ , so we'll have to find both the complementary and the particular solutions!

# Method of Undetermined Coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

To find a particular solution  $y_p$ , we

- ▶ classify the function  $g$  by its type,
- ▶ create an assumed form for the function  $y_p$  (of corresponding type) with unknown coefficients,
- ▶ use substitution into the ODE and matching of like terms to find the coefficients.

We find  $y_c$  using the characteristic equation (do this first), then our general solution

$$y = y_c + y_p.$$

## When part of $g$ solves the homogeneous equation...

$$y'' - y' = 3e^x$$

We tried to find  $y_p$  in the form  $y_p = Ae^x$ , but the process failed. The solution to the associated homogeneous equation

$$y_c = c_1 + c_2 e^x.$$

The problem here is that  $Ae^x$  is the same as  $c_2 e^x$ . The method can still be used, but the form of the particular solution has to be modified.

## We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where  $y_{p_i}$  has the same **general form** as  $g_i(x)$ .

**Case I:**  $y_p$  as first written has no part that duplicates the complementary solution  $y_c$ . Then this first form will suffice.

**Case II:**  $y_p$  has a term  $y_{p_i}$  that duplicates a term in the complementary solution  $y_c$ . Multiply  $y_{p_i}$  by  $x^n$ , where  $n$  is the smallest positive integer that eliminates the duplication.

## Case II Examples

Solve the ODE

$$y'' - 2y' + y = -4e^x$$

Constant coef. left, exponential right.

Find  $y_c$ :  $y'' - 2y' + y = 0$       characteristic eqn

$$m^2 - 2m + 1 = 0$$

$$(m-1)^2 = 0 \Rightarrow m=1 \text{ repeated}$$

$$y_1 = e^x, y_2 = xe^x$$

$$y_c = c_1 e^x + c_2 x e^x$$

Find  $y_p$ :  $y_p(x) = -4e^x$  constant times  $e^x$

1<sup>st</sup>  $y_p = Ae^x$  this is y<sub>1</sub>. won't work

2<sup>nd</sup>  $y_p = Axe^x$  this is y<sub>2</sub> won't work

3<sup>rd</sup>  $y_p = Ax^2e^x$  Should work!

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$

$$y_p'' - 2y_p' + y_p =$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x$$

Collect like terms

$$x^2 e^x (A - 2A + A) + x e^x (4A - 4A) + 2A e^x = -4 e^x$$

$\underbrace{\phantom{0}}_0 \quad \underbrace{\phantom{0}}_0$

$$2A e^x = -4 e^x$$

$$A = -2$$

so  $y_p = -2 x^2 e^x$

The general solution  $y = y_c + y_p$

$$y = C_1 e^x + C_2 x e^x - 2x^2 e^x$$

Find the form of the particular solution

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find  $y_c$ :  $y'' - 4y' + 4y = 0$

$$\begin{aligned}m^2 - 4m + 4 &= 0 \\(m-2)^2 &= 0 \Rightarrow m=2 \text{ repeated}\end{aligned}$$
$$y_1 = e^{2x}, \quad y_2 = xe^{2x}$$

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

To find  $y_p$ , we'll use superposition and consider

$$y'' - 4y' + 4y = \sin(4x) \quad (\text{for } y_{p1})$$

and

$$y'' - 4y' + 4y = xe^{2x} \quad (\text{for } y_{p2})$$

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

$$g_1(x) = \sin(4x) \quad y_p = A \sin(4x) + B \cos(4x) \quad \text{will work}$$

$$g_2(x) = x e^{2x}$$

$$y_p = (Cx+D)e^{2x} = Cx e^{2x} + D e^{2x} \quad \text{won't work}$$

$$y_p = (Cx+D)x e^{2x} = Cx^2 e^{2x} + Dx e^{2x}$$

Still  
won't  
work

$$y_p = (Cx+D)x^2 e^{2x} = Cx^3 e^{2x} + Dx^2 e^{2x}$$

Correct  
form

So

$$y_p = A \sin(4x) + B \cos(4x) + Cx^3 e^{2x} + Dx^2 e^{2x}$$

Find the form of the particular solution

$$y''' - y'' + y' - y = \cos x + x^4$$

Find  $y_c$ :  $y''' - y'' + y' - y = 0$      $m^3 - m^2 + m - 1 = 0$   
 $(m-1)(m^2+1) = 0$   
 $m=1, m^2+1=0 \Rightarrow m=\pm i$   
 $\alpha=0, \beta=1$

$$y_1 = e^{0x}, y_2 = e^{0x} \cos(x), y_3 = e^{0x} \sin(x)$$

$$y_c = c_1 e^{0x} + c_2 \cos x + c_3 \sin x$$

Find  $y_p$ :

$$y''' - y'' + y' - y = \cos x \quad \text{for } y_{p1}$$
$$y''' - y'' + y' - y = x^4 \quad \text{for } y_{p2}$$

$$g_1(x) = \cos x, \quad y_p = A \cos x + B \sin x \quad \text{won't work!}$$

$$y_p = (A \cos x + B \sin x) x$$

$$= Ax \cos x + Bx \sin x \quad \text{will work!}$$

$$g_2(x) = x^4 \quad y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G \quad \text{works}$$

$$y_p = Ax \cos x + Bx \sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G$$

## Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

Find general soln.  $y = y_c + y_p$

Find  $y_c$ :  $y'' - y = 0 \quad m^2 - 1 = 0 \Rightarrow m^2 = 1, \quad m = \pm 1$

$$y_1 = e^x, \quad y_2 = e^{-x}$$

$$y_c = c_1 e^x + c_2 e^{-x}$$

Find  $y_p$ :  $g(x) = 4e^{-x}$

$$y_p = (A e^{-x})x = A x e^{-x}$$

correct form

$$y_p = A x e^{-x}$$

$$y_p' = -A x e^{-x} + A e^{-x}$$

$$y_p'' = Ax e^x - Ae^{-x} - A\dot{e}^{-x} = Ax e^x - 2Ae^{-x}$$

$$y_p'' - y_p =$$

$$Ax e^x - 2Ae^{-x} - Ax e^{-x} = 4e^{-x}$$

$$-2A\dot{e}^{-x} = 4e^{-x}$$

$$A = -2$$

$$\therefore y_p = -2x e^{-x}$$

Gen. Soln  $y = C_1 e^x + C_2 e^{-x} - 2xe^{-x}$

$$y' = C_1 e^x - C_2 e^{-x} - 2e^{-x} + 2xe^{-x}$$

$$y(0) = C_1 e^0 + C_2 e^0 - 2 \cdot 0 e^0 = -1$$

$$y'(0) = C_1 e^0 - C_2 e^0 - 2e^0 + 2 \cdot 0 e^0 = 1$$

$$\begin{aligned} C_1 + C_2 &= -1 \\ C_1 - C_2 - 2 &= 1 \end{aligned} \quad \left. \begin{array}{l} C_1 + C_2 = -1 \\ C_1 - C_2 = 3 \end{array} \right\} \Rightarrow 2C_1 = 2$$

$$C_1 = 1, \quad C_2 = -2$$

Finally, the sol. to the IVP

is

$$y = e^x - 2e^{-x} - 2xe^0$$