March 6 Math 2306 sec. 60 Spring 2018

Section 9: Method of Undetermined Coefficients

The context here is linear, constant coefficient, nonhomogeneous equations

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

where g comes from the restricted classes of functions

- polynomials,
- exponentials,
- sines and/or cosines,
- and products and sums of the above kinds of functions

Recall $y = y_c + y_p$, so we'll have to find both the complementary and the particular solutions!

Method of Undetermined Coefficients

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = g(x)$$

To find a particular solution y_p, we

- classify the function g by its type,
- created an assumed form for the function y_p (of corresponding type) with unknown coefficients,
- use substitution into the ODE and matching of like terms to find the coefficients.

We find y_c using the characteristic equation (do this first), then our general solution

$$y = y_c + y_p.$$

When part of g solves the homogeneous equation...

$$y'' - y' = 3e^x$$

We tried to find y_p in the form $y_p = Ae^x$, but the process failed. The solution to the associated homogeneous equation

$$y_c = c_1 + c_2 e^x.$$

The problem here is that Ae^x is the same as $c_2 e^x$. The method can still be used, but the form of the particular solution has to be modified.

We'll consider cases

Using superposition as needed, begin with assumption:

$$y_p = y_{p_1} + \cdots + y_{p_k}$$

where y_{p_i} has the same **general form** as $g_i(x)$.

Case I: y_p as first written has no part that duplicates the complementary solution y_c. Then this first form will suffice.

Case II: y_p has a term y_{p_i} that duplicates a term in the complementary solution y_c. Multiply y_{p_i} by x^n, where n is the smallest positive integer that eliminates the duplication.

Case II Examples

Solve the ODE

$$y'' - 2y' + y = -4e^x$$

Constant coef. left, exponential right.

Looking for $y = y_c + y_p$.

Find y_c: $y'' - 2y' + y = 0$

Characteristic eqn $m^2 - 2m + 1 = 0$

$(m-1)^2 = 0 \Rightarrow m = 1$ repeated

$y_1 = e^x$, $y_2 = xe^x$

$y_c = c_1 e^x + c_2 x e^x$

Find y_p : $g(x) = -4e^x$

1st Guess $y_p = Ae^x$ won't work, part of y_c

2nd $y_p = (Ae^x)x = Axe^x$ won't work, part of y_c

3rd $y_p = (Axe^x)x = Ax^2e^x$ This will work.

$$y_p' = Ax^2e^x + 2Axe^x$$

$$y_p'' = Ax^2e^x + 2Axe^x + 2Axe^x + 2Ae^x$$

$$= Ax^2e^x + 4Axe^x + 2Ae^x$$

$$y_p'' - 2y_p' + y_p = -4e^x$$

$$Ax^2e^x + 4Axe^x + 2Ae^x - 2(Ax^2e^x + 2Axe^x) + Ax^2e^x = -4e^x$$

Collect like terms

$$x^2e^x(A - 2A + A) + xe^x(4A - 4A) + 2Ae^x = -4e^x$$

$$0 \qquad\qquad 0$$

$$2Ae^x = -4e^x$$

$$A = -2$$

So $y_p = -2x^2e^x$

The general solution $y = y_c + y_p$

$$\boxed{y = c_1 e^x + c_2 x e^x - 2x^2 e^x}$$

Find the form of the particular soluition

$$y'' - 4y' + 4y = \sin(4x) + xe^{2x}$$

Find y_c: $\quad y'' - 4y' + 4y = 0 \qquad m^2 - 4m + 4 = 0$

$(m-2)^2 = 0 \quad m = 2$ repeated

$y_1 = e^{2x}, \quad y_2 = xe^{2x}$

$$y_c = C_1 e^{2x} + C_2 x e^{2x}$$

To find the form of y_p, consider the subproblems

$y'' - 4y' + 4y = \sin(4x)$ for y_{p_1}

and $y'' - 4y' + 4y = xe^{2x}$ for y_{p_2}

$$y_c = c_1 e^{2x} + c_2 x e^{2x}$$

$g_1(x) = \sin(4x)$ $\qquad y_{p_1} = A\sin(4x) + B\cos(4x)$ $\qquad$ will work

$g_2(x) = xe^{2x}$ $\qquad y_{p_2} = (Cx+D)e^{2x} = Cxe^{2x} + De^{2x}$ $\qquad$ Not rsLt

$$y_{p_2} = (Cx+D)xe^{2x} = Cx^2e^{2x} + Dxe^{2x} \qquad \text{Nope!}$$

$$y_{p_2} = (Cx+D)x^2e^{2x} = Cx^3e^{2x} + Dx^2e^{2x} \qquad \text{Yep!}$$

$$\boxed{y_p = A\sin(4x) + B\cos(4x) + Cx^3e^{2x} + Dx^2e^{2x}}$$

Find the form of the particular soluition

$$y''' - y'' + y' - y = \cos x + x^4$$

Find y_c: $y''' - y'' + y' - y = 0$

$m^3 - m^2 + m - 1 = 0$

$(m-1)(m^2+1) = 0$

$m = 1$ or $m^2 + 1 = 0$, $m^2 = -1$

$m = \pm i$

$\alpha = 0$, $\beta = 1$

$y_1 = e^x$, $y_2 = e^{0x}\cos x$, $y_3 = e^{0x}\sin x$

$y_c = c_1 e^x + c_2 \cos x + c_3 \sin x$

To find y_p consider

$y''' - y'' + y' - y = \cos x$ for y_{p_1}

$$y''' - y'' + y' - y = x^4$$ for y_{p2}

$g_1(x) = \cos x$, $\quad y_{p_1} = A\cos x + B\sin x$ won't work

$$y_{p_1} = (A\cos x + B\sin x)x$$

$$= Ax\cos x + Bx\sin x \quad \checkmark$$

$g_2(x) = x^4$ $\quad y_{p_2} = Cx^4 + Dx^3 + Ex^2 + Fx + G \quad \checkmark$

$$\boxed{y_p = Ax\cos x + Bx\sin x + Cx^4 + Dx^3 + Ex^2 + Fx + G}$$

Solve the IVP

$$y'' - y = 4e^{-x} \quad y(0) = -1, \quad y'(0) = 1$$

We need $y = y_c + y_p$.

Find y_c: $y'' - y = 0$ $\quad m^2 - 1 = 0$

$(m-1)(m+1) = 0 \Rightarrow m = 1$ or $m = -1$

$y_1 = e^{x}, \quad y_2 = e^{-x}$

$y_c = c_1 e^{x} + c_2 e^{-x}$

Find y_p: $g(x) = 4e^{-x}$, $\quad y_p = (Ae^{-x})x = Axe^{-x}$

$$y_p = Axe^{-x}$$

$$y_p' = -Axe^{-x} + Ae^{-x}$$

$$y_p'' = Axe^{-x} - Ae^{-x} - Ae^{-x} = Axe^{-x} - 2Ae^{-x}$$

$$y_p'' - y_p = 4e^{-x}$$

$$Axe^{-x} - 2Ae^{-x} - Axe^{-x} = 4e^{-x}$$

$$xe^{-x}\underbrace{(A - A)}_{0} + e^{-x}(-2A) = 4e^{-x}$$

$$-2Ae^{-x} = 4e^{-x}$$

$$A = -2$$

$$y_p = -2xe^{-x}$$

Gen. soln. $y = c_1 e^x + c_2 e^{-x} - 2xe^{-x}$

Apply $y(0) = -1$, $y'(0) = 1$

$y'(x) = c_1 e^x - c_2 e^{-x} - 2e^{-x} + 2xe^{-x}$

$y(0) = c_1 e^0 + c_2 e^0 - 2 \cdot 0 e^0 = -1$

$c_1 + c_2 = -1$

$y'(0) = c_1 e^0 - c_2 e^0 - 2e^0 + 2 \cdot 0 e^0 = 1$

$c_1 - c_2 - 2 = 1$

$c_1 - c_2 = 3$

$$c_1 + c_2 = -1$$

add $$c_1 - c_2 = 3$$

$$2c_1 = 2 \implies c_1 = 1, \quad c_2 = -2$$

$$\boxed{\text{The solution to the IVP is} \quad y = e^{x} - 2e^{-x} - 2xe^{-x}}$$