March 6 Math 3260 sec. 51 Spring 2020

Section 4.3: Linearly Independent Sets and Bases

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solutions $c_1 = c_2 = \cdots = c_p = 0$.
The set is **linearly dependent** if there exist a nontrivial solution (at least one of the weights c_i is nonzero). If there is a nontrivial solution $c_1, \ldots, c_p$, then equation (1) is called a **linear dependence relation**.

Theorem: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ for $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Example

Determine if the set is linearly dependent or independent in $\mathbb{P}_2$.

(a) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = 2t$, $\mathbf{p}_3 = t - 3$.

$$\vec{p}_3 = t - 3 = \frac{1}{2}\vec{p}_2 - 3\vec{p}_1$$

$$\Longrightarrow \quad -3\vec{p}_1 + \frac{1}{2}\vec{p}_2 - \vec{p}_3 = \vec{0}$$

This is a lin. dependence relation with $c_1 = -3$, $c_2 = \frac{1}{2}$, $c_3 = -1$.

The vectors are linearly dependent.

(b) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

Consider $\quad c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{0}$

$$2c_1 + c_2 t - c_3 t^2 = \vec{0}$$
$$= 0 + 0t + 0t^2$$

These have to be equal for all real t.

Letting $t=0$, the equation becomes

$$2c_1 + c_2(0) - c_3\cdot(0^2) = 0 + 0 + 0 = 0$$

$$2c_1 = 0 \implies \boxed{c_1 = 0}$$

Let's let $t=1$ and $t=-1$.

when $t=1$, we set $\quad c_2(1) - c_3(1^2) = 0$

$$c_2 - c_3 = 0 \Rightarrow c_2 = c_3$$

when $t=-1$, we set $\quad c_2(-1) - c_3((-1)^2) = 0$

$$-c_2 - c_3 = 0 \Rightarrow c_2 = -c_3$$

So $c_3 = -c_3 \Rightarrow \boxed{c_3 = 0}$ and $\boxed{c_2 = 0}$ too.

All of $c_1 = c_2 = c_3 = 0$ is the only solution.

This set is linearly independent.

Example

Show that every vector $\mathbf{p} = p_0 + p_1 t + p_2 t^2$ in $\mathbb{P}_2$ can be written as a linear combination of $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$[1] where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

We'll show that we can always find coefficients c_1, c_2, c_3 such that $\vec{p} = c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3$.

$$p_0 + p_1 t + p_2 t^2 = 2c_1 + c_2 t - c_3 t^2$$

Matching coefficients let

$$c_1 = \frac{1}{2}p_0, \quad c_2 = p_1 \quad \text{and} \quad c_3 = -p_2$$

[1] i.e. this set *spans* $\mathbb{P}_2$

For example $\vec{p} = 24 + 2t + 7t^2$

$$\vec{p} = c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3$$
$$= 2c_1 + c_2 t - c_3 t^2$$

This is true if $c_1 = \frac{1}{2}(24) = 12$

$c_2 = 2$, and

$c_3 = -7$

Definition (Basis)

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.