March 6 Math 3260 sec. 55 Spring 2018

Section 4.1: Vector Spaces and Subspaces

Definition A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Examples of Vector Spaces

For an integer $n \geq 0$, $\mathbb{P}_n$ denotes the set of all polynomials with real coefficients of degree at most n. That is

$$\mathbb{P}_n = \{\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \ldots, p_n \in \mathbb{R}\},$$

where addition[1] and scalar multiplication are defined by

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0 + q_0) + (p_1 + q_1)t + \cdots + (p_n + q_n)t^n,$$

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

[1] $\mathbf{q}(t) = q_0 + q_1 t + \cdots + q_n t^n$

$\mathbb{P}_1$

Verify that $\mathbb{P}_1$ is closed under vector addition and scalar multiplication.

A vector in $\mathbb{P}_1$ looks like $\vec{p}(t) = p_0 + p_1 t$, p_0, p_1 in $\mathbb{R}$.

If $\vec{p}, \vec{q}$ are in $\mathbb{P}_1$ then $\vec{p}(t) = p_0 + p_1 t$, $\vec{q}(t) = q_0 + q_1 t$

$$(\vec{p} + \vec{q})(t) = \vec{p}(t) + \vec{q}(t) = (p_0 + q_0) + (p_1 + q_1)t$$

a polynomial of degree at most 1 so it's in $\mathbb{P}_1$.

For scalar c,

$$(c\vec{p})(t) = c\vec{p}(t) = cp_0 + cp_1 t$$

a polynomial of degree at most 1 so it's in $\mathbb{P}_1$

$\mathbb{P}_1$ is closed under vector addition and scaler multiplication as the operations are defined.

Examples of Vector Spaces

Let V be the set of all differentiable, real valued functions $f(x)$ defined for $-\infty < x < \infty$ with the property that

$$f(0) = 0.$$

Define vector addition and scalar multiplication in the standard way for functions—i.e.

$$(f+g)(x) = f(x) + g(x), \quad \text{and} \quad (cf)(x) = cf(x).$$

Example

Verify that properties 1. and 6. hold.

Suppose f and g are in V. Then f and g are differentiable on $(-\infty, \infty)$ and $f(0)=0$ and $g(0)=0$.

$$(f+g)(x) = f(x)+g(x).$$

This is differentiable with domain $(-\infty, \infty)$.

And $(f+g)(0) = f(0)+g(0) = 0+0 = 0$.

Hence $f+g$ is in V which is closed under addition.

For scalar c

$$(cf)(x) = cf(x).$$

This is differentiable with domain $(-\infty, \infty)$.

Also

$$(cf)(0) = cf(0) = c \cdot 0 = 0$$

So cf is in V which is closed under scalar multiplication.

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(1) Does property 1. hold for V?

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ be in V so $x \leq 0, y \leq 0, u \leq 0$ and $v \leq 0$.

$$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x+u \\ y+v \end{bmatrix}$$

$u + x \leq 0$

and

$y + v \leq 0$

The sum is in V. V is closed under vector addition.

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(2) Does property 6. hold for V?

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ be in V so $x \leq 0$ and $y \leq 0$

Is it necessary that $c\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} cx \\ cy \end{bmatrix}$ will satisfy $cx \leq 0$, $cy \leq 0$?

No

Note $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is in V but

$-2\begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is not in V. V is not closed under scalar multiplication.

Theorem

Let V be a vector space. For each $\mathbf{u}$ in V and scalar c

$$\begin{aligned} 0\mathbf{u} &= \mathbf{0} \\ c\mathbf{0} &= \mathbf{0} \\ -1\mathbf{u} &= -\mathbf{u} \end{aligned}$$

Subspaces

Definition: A **subspace** of a vector space V is a subset H of V for which

a) The zero vector is in H[2]

b) H is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in H implies $\mathbf{u} + \mathbf{v}$ is in H)

c) H is closed under scalar multiplication. (i.e. $\mathbf{u}$ in H implies $c\mathbf{u}$ is in H)

[2] This is sometimes replaced with the condition that H is nonempty.

Example

Consider $\mathbb{R}^n$ and let V =Span$\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_p\}$ be a nonempty ($p \geq 1$) subset of $\mathbb{R}^n$. Show that V is a subspace.

We need to show that V satisfies the 3 conditions of a subspace.

① Note $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \dots + 0\vec{v}_p$ i.e. $\vec{0}$ is in Span$\{\vec{v}_1, \dots, \vec{v}_p\}$.

$\vec{0}$ is in V.

② If $\vec{x}, \vec{u}$ are in V then

$\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_p\vec{v}_p$ and

$\vec{u} = a_1\vec{v}_1 + a_2\vec{v}_2 + \dots + a_p\vec{v}_p$ for some scalars $c_1, \dots, c_p$ and $a_1, \dots, a_p$.

$$\vec{x} + \vec{u} = (c_1 + a_1)\vec{v}_1 + (c_2 + a_2)\vec{v}_2 + \cdots + (c_p + a_p)\vec{v}_p$$

Hence $\vec{x} + \vec{u}$ is in V which is closed under vector addition.

③ For scalar k

$$k\vec{x} = kc_1\vec{v}_1 + kc_2\vec{v}_2 + \cdots + kc_p\vec{v}_p$$

Hence $k\vec{x}$ is in V which is closed under scalar multiplication.

V is a subspace of $\mathbb{R}^n$.

Example

Determine which of the following is a subspace of $\mathbb{R}^2$.

(a) The set of all vectors of the form $\mathbf{u} = (u_1, 0)$.

① Note $\vec{0} = (0,0)$ has the right form, it's in this set.

② If $\vec{u} = (u_1, 0)$ and $\vec{v} = (v_1, 0)$, then

$\vec{u} + \vec{v} = (u_1 + v_1, 0 + 0) = (u_1 + v_1, 0)$ has the right form. This set is closed under vector addition.

③ For scalar c, $c\vec{u} = (cu_1, c \cdot 0) = (cu_1, 0)$

The set is closed under scalar multiplication

The set is a subspace of $\mathbb{R}^2$.

Example continued

(b) The set of all vectors of the form $\mathbf{u} = (u_1, 1)$.

This is not a subspace of $\mathbb{R}^2$,

In fact, $\vec{0}$ is not in this set.

Definition: Linear Combination and Span

Definition Let V be a vector space and $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$ be a collection of vectors in V. A **linear combination** of the vectors is a vector $\mathbf{u}$

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

for some scalars $c_1, c_2, \ldots, c_p$.

Definition The **span**, $\text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is the subet of V consisting of all linear combinations of the vectors $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$.

Theorem

Theorem: If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, for $p \geq 1$, are vectors in a vector space V, then Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is a subspace of V.

Remark This is called the **subspace of V spanned by (or generated by)** $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Moreover, if H is any subspace of V, a **spanning set** for H is any set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $H =$Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example

$M^{2\times 2}$ denotes the set of all 2×2 matrices with real entries. Consider the subset H of $M^{2\times 2}$

$$H=\left\{\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix} \mid a,b\in\mathbb{R}\right\}.$$

Show that H is a subspace of $M^{2\times 2}$ by finding a spanning set. That is, show that H =Span$\{\mathbf{v}_1,\mathbf{v}_2\}$ for some appropriate vectors.

Take $\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}$ in H, note

$$\begin{bmatrix} a & 0\\ 0 & b\end{bmatrix}=\begin{bmatrix} a & 0\\ 0 & 0\end{bmatrix}+\begin{bmatrix} 0 & 0\\ 0 & b\end{bmatrix}=\underbrace{a\begin{bmatrix} 1 & 0\\ 0 & 0\end{bmatrix}+b\begin{bmatrix} 0 & 0\\ 0 & 1\end{bmatrix}}_{\text{linear combination of constant vector}}$$

Let $\vec{v}_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$.

$$H = \mathrm{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}.$$