March 6 Math 3260 sec. 55 Spring 2020

Section 4.3: Linearly Independent Sets and Bases

Definition: A set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ in a vector space V is said to be **linearly independent** if the equation

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p = \mathbf{0} \tag{1}$$

has only the trivial solutions $c_1 = c_2 = \cdots = c_p = 0$.
The set is **linearly dependent** if there exist a nontrivial solution (at least one of the weights c_i is nonzero). If there is a nontrivial solution $c_1, \ldots, c_p$, then equation (1) is called a **linear dependence relation.**

Theorem: The set $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$, $p \geq 2$ and $\mathbf{v}_1 \neq \mathbf{0}$, is linearly dependent if and only if some $\mathbf{v}_j$ for $j > 1$ is a linear combination of the preceding vectors $\mathbf{v}_1, \ldots, \mathbf{v}_{j-1}$.

Example

Determine if the set is linearly dependent or independent in $\mathbb{P}_2$.

(a) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 1$, $\mathbf{p}_2 = 2t$, $\mathbf{p}_3 = t - 3$.

$$\vec{p}_3 = t - 3 = \tfrac{1}{2}\vec{p}_2 - 3\vec{p}_1$$

$$\Rightarrow \quad \tfrac{1}{2}p_2 - 3\vec{p}_1 - \vec{p}_3 = \vec{0}$$

$$-3\vec{p}_1 + \tfrac{1}{2}\vec{p}_2 - \vec{p}_3 = \vec{0}$$

This is a linear dependence relation,

the set is linearly dependent.

(b) $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

Consider the equation

$$c_1\vec{p}_1 + c_2\vec{p}_2 + c_3\vec{p}_3 = \vec{0}$$

$$2c_1 + c_2 t - c_3 t^2 = 0 + 0t + 0t^2$$

This is supposed to hold for all real numbers t,

When $t=0$, the equation becomes

$$2c_1 + c_2(0) - c_3(0^2) = 0 + 0 + 0 = 0$$

$$2c_1 = 0$$

$$\boxed{c_1 = 0}$$

When $t=1$, the equation becomes

$$c_2(1) - c_3(1^2) = 0 \Rightarrow c_2 - c_3 = 0$$
$$c_2 = c_3$$

When $t=-1$, we get

$$c_2(-1) - c_3(-1)^2 = 0 \Rightarrow -c_2 - c_3 = 0$$
$$c_2 = -c_3$$

$c_3 = -c_3 \Rightarrow$ $c_3 = 0$ so $c_2 = 0$

The only solution is $c_1 = c_2 = c_3 = 0$.

The set is linearly independent.

Example

Show that every vector $\mathbf{p} = p_0 + p_1 t + p_2 t^2$ in $\mathbb{P}_2$ can be written as a linear combination of $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$[1] where $\mathbf{p}_1 = 2,\ \mathbf{p}_2 = t,\ \mathbf{p}_3 = -t^2$.

we want $\vec{p} = c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3$

$$p_0 + p_1 t + p_2 t^2 = 2 c_1 + c_2 t - c_3 t^2$$

This holds if $c_1 = \frac{1}{2} p_0$, $c_2 = p_1$, and $c_3 = -p_2$

$$c_1 \vec{p}_1 + c_2 \vec{p}_2 + c_3 \vec{p}_3 = 2\left(\tfrac{1}{2} p_0\right) + p_1 t - (-p_2) t^2$$

$$= p_0 + p_1 t + p_2 t^2$$

[1] i.e. this set *spans* $\mathbb{P}_2$

Definition (Basis)

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

Example

If A is an invertible $n \times n$ matrix, then we know[2] that (1) the columns are linearly independent, and (2) the columns span $\mathbb{R}^n$. Use this to determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

We can create a matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$.

If A is invertible, the set is a basis, otherwise it's not.

[2]from our large theorem on invertible matrices from section 2.3

A is invertible if $\operatorname{rref} A = I$

$$\begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Yes, A^{-1} exists so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Standard Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$. For example, the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{respectively.}$$