March 6 Math 3260 sec. 56 Spring 2018

Section 4.1: Vector Spaces and Subspaces

Definition A **vector space** is a nonempty set V of objects called *vectors* together with two operations called *vector addition* and *scalar multiplication* that satisfy the following ten axioms: For all $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ in V, and for any scalars c and d

1. The sum $\mathbf{u}+\mathbf{v}$ of $\mathbf{u}$ and $\mathbf{v}$ is in V.
2. $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
3. $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
4. There exists a **zero** vector $\mathbf{0}$ in V such that $\mathbf{u}+\mathbf{0}=\mathbf{u}$.
5. For each vector $\mathbf{u}$ there exists a vector $-\mathbf{u}$ such that $\mathbf{u}+(-\mathbf{u})=\mathbf{0}$.
6. For each scalar c, $c\mathbf{u}$ is in V.
7. $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
8. $(c+d)\mathbf{u}=c\mathbf{u}+d\mathbf{u}$.
9. $c(d\mathbf{u})=d(c\mathbf{u})=(cd)\mathbf{u}$.
10. $1\mathbf{u}=\mathbf{u}$

Examples of Vector Spaces

For an integer $n \geq 0$, $\mathbb{P}_n$ denotes the set of all polynomials with real coefficients of degree at most n. That is

$$\mathbb{P}_n = \{\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n \mid p_0, p_1, \ldots, p_n \in \mathbb{R}\},$$

where addition[1] and scalar multiplication are defined by

$$(\mathbf{p} + \mathbf{q})(t) = \mathbf{p}(t) + \mathbf{q}(t) = (p_0 + q_0) + (p_1 + q_1)t + \cdots + (p_n + q_n)t^n,$$

$$(c\mathbf{p})(t) = c\mathbf{p}(t) = cp_0 + cp_1 t + \cdots + cp_n t^n.$$

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[1] $\mathbf{q}(t) = q_0 + q_1 t + \cdots + q_n t^n$

Example

We found that in $\mathbb{P}_n$, the zero vector $\mathbf{0}(t) = 0 + 0t + \cdots + 0t^n$.

If $\mathbf{p}(t) = p_0 + p_1 t + \cdots + p_n t^n$, what is the vector $-\mathbf{p}$?

for $-\vec{p}(t) = a_0 + a_1 t + \ldots + a_n t^n$

$$(\vec{p} + (-\vec{p}))(t) = \vec{p}(t) + (-\vec{p}(t))$$

$$= (p_0 + a_0) + (p_1 + a_1)t + \ldots + (p_n + a_n)t^n$$

$$= 0 \quad + \quad 0t \quad + \quad \ldots \quad + \quad 0t^n$$

$a_0 = -p_0, \quad a_1 = -p_1, \quad \ldots, \quad a_n = -p_n$

so $-\vec{p}(t) = -p_0 - p_1 t - \ldots - p_n t^n$.

Examples of Vector Spaces

Let V be the set of all differentiable, real valued functions $f(x)$ defined for $-\infty < x < \infty$ with the property that

$$f(0) = 0.$$

Define vector addition and scalar multiplication in the standard way for functions—i.e.

$$(f+g)(x) = f(x) + g(x), \quad \text{and} \quad (cf)(x) = cf(x).$$

Example

Verify that properties 1. and 6. hold.

Let f, g be in V so f and g are differentiable with domain $(-\infty, \infty)$, $f(0)=0$ and $g(0)=0$.

$(f+g)(x) = f(x) + g(x)$ is differentiable with domain $(-\infty,\infty)$.

$(f+g)(0) = f(0) + g(0) = 0 + 0 = 0$

So $f+g$ is in V.

V is closed under vector addition.

For scalar k

$(kf)(x) = kf(x)$ is differentiable with domain $(-\infty, \infty)$.

$$(kf)(0) = kf(0) = k \cdot 0 = 0$$

So V is closed under scaler multiplication.

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(1) Does property 1. hold for V?

Let $\begin{bmatrix} x \\ y \end{bmatrix}$ and $\begin{bmatrix} u \\ v \end{bmatrix}$ be in V so $x \leq 0$, $y \leq 0$, $u \leq 0$, and $v \leq 0$.

$\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x+u \\ y+v \end{bmatrix}$ Note $x \leq 0$ and $u \leq 0$

$\Rightarrow x+u \leq 0$

Similarly $y+v \leq 0$.

The sum is in V.

V is closed under vector addition.

A set that is not a Vector Space

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, \mid x \leq 0, y \leq 0 \right\}$ with regular vector addition and scalar multiplication in $\mathbb{R}^2$. Note V is the third quadrant in the xy-plane.

(2) Does property 6. hold for V?

If $\begin{bmatrix} x \\ y \end{bmatrix}$ is in V and, say $x < 0$, then note that

$-2 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -2x \\ -2y \end{bmatrix}$ is not in V since $-2x > 0$.

For example, $\begin{bmatrix} -1 \\ -1 \end{bmatrix}$ is in V, but $-2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$ is not in V.

V is not closed under scalar multiplication. V is not a Vector space.

Theorem

Let V be a vector space. For each $\mathbf{u}$ in V and scalar c

$$\begin{aligned} 0\mathbf{u} &= \mathbf{0} \\ c\mathbf{0} &= \mathbf{0} \\ -1\mathbf{u} &= -\mathbf{u} \end{aligned}$$

Subspaces

Definition: A **subspace** of a vector space V is a subset H of V for which

a) The zero vector is in H[2]

b) H is closed under vector addition. (i.e. $\mathbf{u}, \mathbf{v}$ in H implies $\mathbf{u}+\mathbf{v}$ is in H)

c) H is closed under scalar multiplication. (i.e. $\mathbf{u}$ in H implies $c\mathbf{u}$ is in H)

[2] This is sometimes replaced with the condition that H is nonempty.

Example

Consider $\mathbb{R}^n$ and let $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a nonempty ($p \geq 1$) subset of $\mathbb{R}^n$. Show that V is a subspace.

We'll show that V satisfies the 3 properties.

1. $\vec{0} = 0\vec{v}_1 + 0\vec{v}_2 + \cdots + 0\vec{v}_p$, $\vec{0}$ is in V.

2. Suppose $\vec{x}, \vec{u}$ are in V.

$\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_p\vec{v}_p$ and

$\vec{u} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_p\vec{v}_p$ for some scalars

$a_1, \ldots, a_p, b_1, \ldots, b_p$.

$\vec{x} + \vec{u} = (a_1+b_1)\vec{v}_1 + (a_2+b_2)\vec{v}_2 + \cdots + (a_p+b_p)\vec{v}_p$

The sum is a linear combination of $\vec{v}_1, \dots, \vec{v}_p$.

So the sum is in V. V is closed under vector addition.

③ For scaler c

$$c\vec{x} = ca_1\vec{v}_1 + ca_2\vec{v}_2 + \cdots + ca_p\vec{v}_p$$

a linear combination of $\vec{v}_1, \dots, \vec{v}_p$, hence in V.

V is closed under scaler multiplication.

V is a subspace of $\mathbb{R}^n$.

Example

Determine which of the following is a subspace of $\mathbb{R}^2$.

(a) The set of all vectors of the form $\mathbf{u} = (u_1, 0)$.

① $\vec{0} = (0,0)$ is in this set.

② If $\vec{u} = (u_1, 0)$ and $\vec{v} = (v_1, 0)$,

$$\vec{u} + \vec{v} = (u_1 + v_1, 0 + 0) = (u_1 + v_1, 0)$$

The set is closed under vector addition.

③ $c\vec{u} = (cu_1, c0) = (cu_1, 0)$ The set is closed under scalar multiplication.

The set is a subspace of $\mathbb{R}^2$.

Example continued

(b) The set of all vectors of the form $\mathbf{u} = (u_1, 1)$.

Not a subspace. In particular $\vec{0}$ is not in this set.

Definition: Linear Combination and Span

Definition Let V be a vector space and $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$ be a collection of vectors in V. A **linear combination** of the vectors is a vector $\mathbf{u}$

$$\mathbf{u} = c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \cdots + c_p\mathbf{v}_p$$

for some scalars $c_1, c_2, \ldots, c_p$.

Definition The **span**, $\mathrm{Span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is the subet of V consisting of all linear combinations of the vectors $\mathbf{v}_1$, $\mathbf{v}_2, \ldots, \mathbf{v}_p$.

Theorem

Theorem: If $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p$, for $p \geq 1$, are vectors in a vector space V, then Span$\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$, is a subspace of V.

Remark This is called the **subspace of V spanned by (or generated by)** $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$. Moreover, if H is any subspace of V, a **spanning set** for H is any set of vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$ such that $H =$Span$\{\mathbf{v}_1, \ldots, \mathbf{v}_p\}$.

Example

$M^{2\times 2}$ denotes the set of all 2×2 matrices with real entries. Consider the subset H of $M^{2\times 2}$

$$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

Show that H is a subspace of $M^{2\times 2}$ by finding a spanning set. That is, show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$ for some appropriate vectors.

Let $\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ be in H. Note that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix}$$

$$= a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

linear combination of fixed vectors

So $H = \text{Span}\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$.