March 7 Math 1190 sec. 63 Spring 2017

Section 3.3: Derivatives of Logarithmic Functions

Recall: If $a > 0$ and $a \neq 1$, we denote the **base** a **logarithm** of x by

$$\log_a x$$

This is the inverse function of the (one to one) function $y = a^x$. So we can define $\log_a x$ by the statement

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

Properties of Logarithms

Additional properties that we need to know:

- $f(x) = \log_a(x)$, has domain $(0, \infty)$ and range $(-\infty, \infty)$.
- For $a > 1$, *

$$\lim_{x \to 0^+} \log_a(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} \log_a(x) = \infty$$

- In particular

$$\lim_{x \to 0^+} \ln(x) = -\infty \quad \text{and} \quad \lim_{x \to \infty} \ln(x) = \infty$$

Note that for $0 < a < 1$ we can use the fact that

$$\log_{\frac{1}{b}}(x) = -\log_b(x).$$

Graphs of Logarithms:Logarithms are differentiable on $(0, \infty)$.

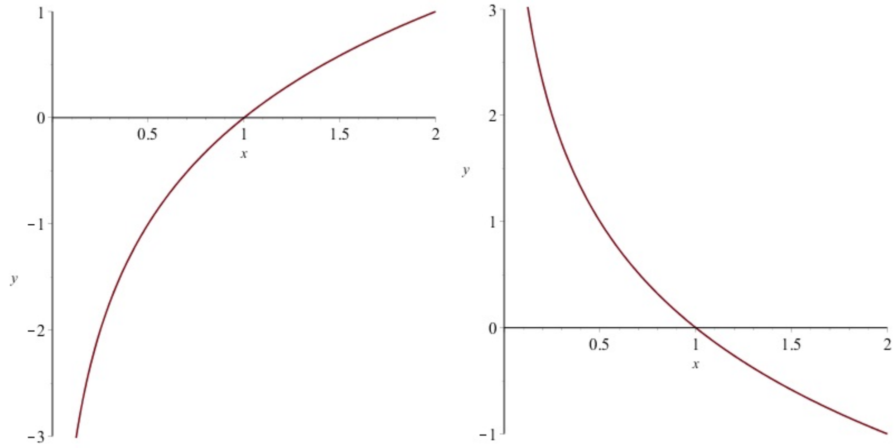

Figure: Plots of functions of the type $f(x) = \log_a(x)$. The value of $a > 1$ on the left, and $0 < a < 1$ on the right.

The Derivative of $y = \log_a(x)$

To find a derivative rule for $y = \log_a(x)$, we use the chain rule.

Let $y = \log_a(x)$, then $x = a^y$.

Using implicit diff: $\frac{d}{dx} x = \frac{d}{dx} a^y$

$$1 = a^y(\ln a)\frac{dy}{dx}$$

Solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{1}{a^y \ln a}$ but $a^y = x$

so $\frac{dy}{dx} = \frac{1}{x \ln a}$

New rule:

$$\boxed{\frac{d}{dx}\log_a x = \frac{1}{x\ln a}}$$

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln(a)}$$

Examples: Evaluate each derivative.

(a) $\frac{d}{dx}\log_3(x) = \frac{1}{x\ln 3}$ here $a=3$

(b) $\frac{d}{d\theta}\log_{\frac{1}{2}}(\theta) = \frac{1}{\theta\ln\frac{1}{2}}$ here $a=\frac{1}{2}$

Question

True or False The derivative of the natural log

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

$$\frac{d}{dx}\ln x = \frac{1}{x\ln e} \quad \text{but} \quad \ln e = 1$$

The function $\ln|x|$

Show that if $x < 0$, then

$$\frac{d}{dx}\ln(-x) = \frac{1}{x}$$

If $x<0$, then $-x>0$.

inside $u=-x$, $u'=-1$

outside $f(u)=\ln u$

$f'(u)=\frac{1}{u}$

$$\frac{d}{dx}\ln(-x)=\frac{1}{(-x)}\cdot(-1)$$

$$=\frac{-1}{-x}=\frac{1}{x}$$

The Derivative of $\ln|x|$

$$\frac{d}{dx}\ln|x| = \frac{1}{x}$$

Differentiating Functions Involving Logs

We can combine our new rule with our existing derivative rules.

Chain Rule: Let u be a differentiable function. Then

$$\frac{d}{dx}\log_a|u| = \frac{1}{u\ln(a)}\frac{du}{dx} = \frac{u'(x)}{u(x)\ln(a)}.$$

In particular

$$\frac{d}{dx}\ln|u| = \frac{1}{u}\frac{du}{dx} = \frac{u'(x)}{u(x)}.$$

i.e. $\quad \frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}$

$$\frac{d}{dx}\ln|f(x)| = \frac{f'(x)}{f(x)}$$

Examples

Evaluate each derivative.

(a) $\displaystyle \frac{d}{dx}\ln|\sec x|$ $\displaystyle = \frac{\sec x\tan x}{\sec x} = \tan x$

(b) $\displaystyle \frac{d}{dt}\, t^2\log_3(4t)$ $\displaystyle = 2t\log_3(4t) + \cancel{t^2}\left(\frac{1}{\cancel{4t}\ln 3}\cdot\cancel{4}\right)$

product

$\displaystyle = 2t\log_3(4t) + \frac{t}{\ln 3}$

Question

Find y' if $y = x\,(\ln x)^2$.

(a) $y' = \frac{2\ln x}{x}$

(b) $y' = 2\ln x + 2$

(c) $y' = (\ln x)^2 + 2\ln x$

(d) $y' = \ln(x^2) + 2$

$$y' = 1 \cdot (\ln x)^2 + x\left[2(\ln x) \cdot \frac{1}{x}\right]$$

$$= (\ln x)^2 + 2\ln x$$

Example

Determine $\frac{dy}{dx}$ if $\quad x \ln y + y \ln x = 10.$

Implicit diff

$$\frac{d}{dx}\left(x \ln y + y \ln x\right) = \frac{d}{dx}(10)$$

$$\left(\frac{d}{dx}x\right)\ln y + x\left(\frac{d}{dx}\ln y\right) + \left(\frac{d}{dx}y\right)\ln x + y\left(\frac{d}{dx}\ln x\right) = 0$$

$$1 \cdot \ln y + x\,\frac{1}{y}\cdot\frac{dy}{dx} + \frac{dy}{dx}\ln x + y\cdot\frac{1}{x} = 0$$

$$\ln y + \frac{x}{y}\frac{dy}{dx} + (\ln x)\frac{dy}{dx} + \frac{y}{x} = 0$$

$$\frac{x}{y}\frac{dy}{dx} + (\ln x)\frac{dy}{dx} = \frac{-y}{x} - \ln y$$

Clean fractions mult by xy

$$xy\left(\frac{x}{y}\frac{dy}{dx}+(\ln x)\frac{dy}{dx}\right)=xy\left(\frac{-y}{x}-\ln y\right)$$

$$x^2\frac{dy}{dx}+xy(\ln x)\frac{dy}{dx}=-y^2-xy\ln y$$

$$(x^2+xy\ln x)\frac{dy}{dx}=-y^2-xy\ln y$$

$$\boxed{\frac{dy}{dx}=\frac{-y^2-xy\ln y}{x^2+xy\ln x}}$$

Using Properties of Logs

Properties of logarithms can be used to simplify expressions characterized by products, quotients and powers.

Illustrative Example: Evaluate $\dfrac{d}{dx}\ln\left(\dfrac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right)$

We use log properties first, then take $\frac{d}{dx}$:

$$\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = \ln\left(x^2\cos(2x)\right) - \ln\left(\sqrt[3]{x^2+x}\right)$$

$$= \ln x^2 + \ln\cos(2x) - \ln\left((x^2+x)^{1/3}\right)$$

$$= 2\ln x + \ln \cos(2x) - \frac{1}{3}\ln(x^2+x)$$

$$\frac{d}{dx}\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = \frac{d}{dx}\left(2\ln x + \ln\cos(2x) - \frac{1}{3}\ln(x^2+x)\right)$$

$$= 2\cdot\frac{1}{x} + \frac{-\sin(2x)\cdot 2}{\cos(2x)} - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$

$$= \frac{2}{x} - \frac{2\sin(2x)}{\cos(2x)} - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$

$$= \frac{2}{x} - 2\tan(2x) - \frac{1}{3}\,\frac{2x+1}{x^2+x}$$