

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these. We require another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x), y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1 , y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have 2 unknowns
 u_1, u_2 but only
one equation, the
DE.

Sub into the DE

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$$

We'll introduce a 2nd
equation for the
 u' 's.

$$\text{We'll assume } u_1' y_1 + u_2' y_2 = 0$$

Remember that $y_i'' + P(x)y_i' + Q(x)y_i = 0$, for $i = 1, 2$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$$

$$y_p'' + P(x)y_p' + Q(x)y_p = g(x)$$

$$\underline{u_1} \underline{y_1''} + \underline{u_1'} \underline{y_1'} + \underline{u_2} \underline{y_2''} + \underline{u_2'} \underline{y_2'} + P(x) \left(\underline{u_1} \underline{y_1'} + \underline{u_2} \underline{y_2'} \right) + Q(x) \left(\underline{u_1} \underline{y_1} + \underline{u_2} \underline{y_2} \right) = g(x)$$

Collect by u 's

$$u_1 \underbrace{(y_1'' + P(x)y_1' + Q(x)y_1)}_{0''} + u_2 \underbrace{(y_2'' + P(x)y_2' + Q(x)y_2)}_{0''} + u_1' y_1' + u_2' y_2' = g(x)$$

Since y_1, y_2 solve the
homogeneous equation

We have 2 equations

$$u_1' y_1 + u_2' y_2 = 0$$

$$u_1' y_1' + u_2' y_2' = g(x)$$

In matrix format

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$



Note
the Wronskian
matrix

Since y_1, y_2 is a
fundamental solution
set, $W(y_1, y_2) \neq 0$

To use Crammer's rule

$$\text{Let } W_1 = \det \begin{pmatrix} 0 & y_2 \\ g & y_2' \end{pmatrix} = -g(x)y_2(x)$$

$$\text{Let } W_2 = \det \begin{pmatrix} y_1 & 0 \\ y_1' & g \end{pmatrix} = g(x)y_1(x)$$

If $W = \det \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}$, the Wronskian

Then $u_1' = \frac{w_1}{w} = \frac{-g(x)y_2(x)}{w}$

and

$$u_2' = \frac{w_2}{w} = \frac{g(x)y_1(x)}{w}$$

so

$$u_1 = \int -\frac{g(x)y_2(x)}{w} dx$$

and

$$u_2 = \int \frac{g(x)y_1(x)}{w} dx$$

Example:

Solve the ODE $y'' + y = \tan x$.

Get y_c : $y'' + y = 0$

$$m^2 + 1 = 0 \Rightarrow m^2 = -1$$

$$m = \pm i = 0 \pm 1i$$

$$\alpha = 0, \beta = 1$$

$$y_1 = \cos x, y_2 = \sin x$$

Find y_p : Let $y_p = u_1(x)y_1 + u_2(x)y_2$

Here $g(x) = \tan x$

$$W(y_1, y_2)(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$$u_1 = \int -\frac{g(x)y_{z(x)}}{w} dx = \int -\frac{\tan x \sin x}{1} dx$$

$$= - \int \tan x \sin x dx = - \int \frac{\sin x}{\cos x} \sin x dx$$

$$= - \int \frac{\sin^2 x}{\cos x} dx = - \int \frac{1 - \cos^2 x}{\cos x} dx$$

$$= \int \frac{\cos^2 x - 1}{\cos x} dx$$

$$= \int (\cos x - \sec x) dx$$

$$u_1 = \sin x - \ln |\sec x + \tan x|$$

$$u_2 = \int \frac{g(x) y_1(x)}{w} dx = \int \frac{\tan x \cos x}{1} dx$$

$$= \int \tan x \cos x dx = \int \frac{\sin x}{\cos x} \cos x dx$$

$$= \int \sin x dx = -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2 =$$

$$= (\sin x - \ln |\sec x + \tan x|) \cos x + (-\cos x) \sin x$$

$$= \sin x \cos x - \cos x \ln |\sec x + \tan x| - \cos x \sin x$$

$$y_p = -\cos x \ln |\sec x + \tan x|$$

The general solution

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln |\sec x + \tan x|$$

Example:

Solve the ODE

$$x^2 y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1 x^2 + c_2 x^{-2}$ is the complementary solution.

We're given $y_1 = x^2$, $y_2 = x^{-2}$.

Standard form

$$y'' + \frac{1}{x} y' - \frac{4}{x^2} y = \frac{\ln x}{x^2}$$

$$g(x) = \frac{\ln x}{x^2}$$

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = -2x^2 x^{-3} - 2x x^{-2} = -4x^{-1}$$

$$u_1 = \int \frac{-g_1(x) y_2(x)}{w} dx = \int -\frac{\ln x}{x^2} (x^{-2}) dx$$

$$= \frac{1}{4} \int (\ln x) x^{-2} (x^{-2}) x dx$$

$$= \frac{1}{4} \int x^{-3} \ln x dx$$

$$= \frac{1}{4} \left(-\frac{x^{-2}}{2} \ln x - \int \frac{x^{-2}}{-2} \cdot \frac{1}{x} dx \right)$$

$$= \frac{1}{4} \left(-\frac{x^{-2}}{2} \ln x + \frac{1}{2} \int x^{-3} dx \right)$$

I by parts

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$v = \frac{x^{-2}}{-2}$$

$$dv = x^{-3} dx$$

$$= \frac{1}{4} \left(-\frac{\ln x}{2x^2} - \frac{1}{2} \cdot \frac{1}{2} x^{-2} \right)$$

$$= -\frac{\ln x}{8x^2} - \frac{1}{16x^2}$$

$$u_2 = \int \frac{g(x) y_1(x)}{w} dx = \int \frac{\frac{\ln x}{x^2} \cdot x^2}{-4x^{-1}} dx$$

$$= -\frac{1}{4} \int x \ln x dx$$

By parts

$$u = \ln x \quad du = \frac{1}{x} dx$$

$$= \frac{1}{4} \left(\frac{x^2}{2} \ln x - \int \frac{x^2}{2} \cdot \frac{1}{x} dx \right)$$

$$v = \frac{x^2}{2} \quad dv = x dx$$

$$= -\frac{1}{4} \left(\frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx \right)$$

$$= -\frac{1}{4} \left(\frac{x^2}{2} \ln x - \frac{1}{6} x^3 \right) = -\frac{x^2 \ln x}{8} + \frac{x^3}{16}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(-\frac{\ln x}{8x^2} - \frac{1}{16x^2} \right) x^2 + \left(\frac{-x^2 \ln x}{8} + \frac{x^2}{16} \right) x^{-2}$$

$$= -\frac{\ln x}{8} - \frac{1}{16} - \frac{\ln x}{8} + \frac{1}{16} = -\frac{\ln x}{4}$$

The general solution

$$y = C_1 x^2 + C_2 \dot{x}^2 - \frac{1}{4} \ln x$$