March 8 Math 2306 sec. 60 Spring 2018

Section 10: Variation of Parameters

We are still considering nonhomogeneous, linear ODEs. Consider equations of the form

$$y'' + y = \tan x, \quad \text{or} \quad x^2y'' + xy' - 4y = e^x.$$

The method of undetermined coefficients is not applicable to either of these. We require another approach.

Variation of Parameters

For the equation in standard form

$$\frac{d^2y}{dx^2} + P(x)\frac{dy}{dx} + Q(x)y = g(x),$$

suppose $\{y_1(x),\ y_2(x)\}$ is a fundamental solution set for the associated homogeneous equation. We seek a particular solution of the form

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$$

where u_1 and u_2 are functions we will determine (in terms of y_1, y_2 and g).

This method is called **variation of parameters**.

Variation of Parameters: Derivation of y_p

$$y'' + P(x)y' + Q(x)y = g(x)$$

Set $\quad y_p = u_1(x)y_1(x) + u_2(x)y_2(x)$

We have 2 unknowns u_1, u_2, one equation, the ODE. We will introduce a 2nd equation in the process.

$y_p = u_1 y_1 + u_2 y_2$

$y_p' = u_1 y_1' + u_2 y_2' + u_1' y_1 + u_2' y_2$

we'll assume $u_1' y_1 + u_2' y_2 = 0$

$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$

Remember that $\quad y_i'' + P(x)y_i' + Q(x)y_i = 0, \quad$ for $i = 1, 2$

$$y_p = u_1 y_1 + u_2 y_2$$

$$y_p' = u_1 y_1' + u_2 y_2'$$

$$y_p'' = u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2'$$

$$y_p'' + P(x) y_p' + Q(x) y_p = g(x)$$

$$u_1 y_1'' + u_2 y_2'' + u_1' y_1' + u_2' y_2' + P(x)\left(u_1 y_1' + u_2 y_2'\right) + Q(x)\left(u_1 y_1 + u_2 y_2\right) = g(x)$$

Collect by u's

$$u_1 \underbrace{\left(y_1'' + P(x) y_1' + Q(x) y_1\right)}_{=0} + u_2 \underbrace{\left(y_2'' + P(x) y_2' + Q(x) y_2\right)}_{=0} + u_1' y_1' + u_2' y_2' = g(x)$$

Since y_1, y_2 solve the homogeneous eqn.

We have 2 eqns for u_1' and u_2'

$$u_1' y_1 + u_2' y_2 = 0$$
$$u_1' y_1' + u_2' y_2' = g$$

In matrix format

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}$$

Using Cramer's rule

Let $W_1 = \begin{vmatrix} 0 & y_2 \\ g & y_2' \end{vmatrix} = -g y_2$

$$W_2 = \begin{vmatrix} y_1 & 0 \\ y_1' & g \end{vmatrix} = g y_1$$

Letting $W = W(y_1, y_2)(x)$ the Wronskian

$$u_1' = \frac{W_1}{W} = \frac{-g y_2}{W}$$

and

$$u_2' = \frac{W_2}{W} = \frac{g y_1}{W}$$

$$u_1 = \int \frac{-g(x)\,y_2(x)}{W}\,dx\,, \quad u_2 = \int \frac{g(x)\,y_1(x)}{W}\,dx$$

Example:

Solve the ODE $y'' + y = \tan x$.

Find y_c : solve $y'' + y = 0$

$m^2 + 1 = 0 \Rightarrow m^2 = -1 \Rightarrow m = \pm i$

$m = 0 \pm 1i \qquad \alpha = 0, \ \beta = 1$

$y_1 = \cos x \qquad y_2 = \sin x$

$g(x) = \tan x$

$$W = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix} = \cos^2 x + \sin^2 x = 1$$

$y_p = u_1 y_1 + u_2 y_2$

$$u_1 = \int \frac{-g(x)\, y_2(x)}{W}\, dx = \int \frac{-\tan x \sin x}{1}\, dx$$

$$= -\int \tan x \sin x \, dx = -\int \frac{\sin x}{\cos x} \sin x \, dx$$

$$= -\int \frac{\sin^2 x}{\cos x}\, dx = -\int \frac{1-\cos^2 x}{\cos x}\, dx$$

$$= \int \frac{\cos^2 x - 1}{\cos x}\, dx = \int (\cos x - \sec x)\, dx$$

$$= \sin x - \ln|\sec x + \tan x|$$

$$u_2 = \int \frac{g(x)\, y_1(x)}{W}\, dx = \int \frac{\tan x \, \cos x}{1}\, dx$$

$$= \int \tan x \cos x \, dx = \int \frac{\sin x}{\cos x} \cos x \, dx$$

$$= \int \sin x \, dx$$

$$= -\cos x$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\sin x - \ln|\sec x + \tan x|\right)\cos x + (-\cos x)\sin x$$

$$= \sin x \cos x - \cos x \ln|\sec x + \tan x| - \cos x \sin x$$

$$y_p = -\cos x \ln|\sec x + \tan x|$$

The general solution to the ODE

$$y = C_1 \cos x + C_2 \sin x - \cos x \ln|\sec x + \tan x|$$

Example:

Solve the ODE

$$x^2y'' + xy' - 4y = \ln x,$$

given that $y_c = c_1x^2 + c_2x^{-2}$ is the complementary solution.

$y_1 = x^2, \quad y_2 = x^{-2}$

Standard form $\quad y'' + \frac{1}{x}y' - \frac{4}{x^2}y = \frac{\ln x}{x^2}$

$g(x) = \frac{\ln x}{x^2}$

$$W = \begin{vmatrix} x^2 & x^{-2} \\ 2x & -2x^{-3} \end{vmatrix} = x^2(-2x^{-3}) - 2x(x^{-2}) = -4x^{-1}$$

$$u_1 = \int \frac{-g(x)\,y_2(x)}{W}\,dx = \int \frac{-\frac{\ln x}{x^2}\,x^{-2}}{-4x^{-1}}\,dx$$

$$= \frac{1}{4}\int (\ln x)\,x^{-2}\cdot x^{-2}\cdot x\,dx$$

$$= \frac{1}{4}\int x^{-3}\ln x\,dx$$

Int by parts

$u = \ln x \qquad du = \frac{1}{x}\,dx$

$v = \frac{x^{-2}}{-2} \qquad dv = x^{-3}\,dx$

$$= \frac{1}{4}\left(\frac{x^{-2}}{-2}\ln x - \int \frac{x^{-2}}{-2}\cdot\frac{1}{x}\,dx\right)$$

$$= \frac{1}{4}\left(\frac{-\ln x}{2x^2} + \frac{1}{2}\int x^{-3}\,dx\right)$$

$$=\frac{1}{4}\left(\frac{-\ln x}{2x^2}-\frac{1}{4}x^{-2}\right)=\frac{-\ln x}{8x^2}-\frac{1}{16x^2}$$

$$u_2=\int\frac{g(x)y_1(x)}{W}\,dx=\int\frac{\frac{\ln x}{x^2}\cdot x^2}{-4x^{-1}}\,dx$$

$$=\frac{-1}{4}\int x\ln x\,dx$$

Int by parts

$u=\ln x \qquad du=\frac{1}{x}\,dx$

$v=\frac{x^2}{2} \qquad dv=x\,dx$

$$=\frac{-1}{4}\left(\frac{x^2}{2}\ln x-\int\frac{x^2}{2}\frac{1}{x}\,dx\right)$$

$$=\frac{-1}{4}\left(\frac{x^2}{2}\ln x-\frac{1}{2}\int x\,dx\right)$$

$$= -\frac{1}{4}\left(\frac{x^2}{2}\ln x - \frac{x^2}{4}\right) = \frac{-x^2\ln x}{8} + \frac{x^2}{16}$$

$$y_p = u_1 y_1 + u_2 y_2$$

$$= \left(\frac{-\ln x}{8x^2} - \frac{1}{16x^2}\right)x^2 + \left(\frac{-x^2\ln x}{8} + \frac{x^2}{16}\right)x^{-2}$$

$$= \frac{-\ln x}{8} - \frac{1}{16} - \frac{\ln x}{8} + \frac{1}{16} = \frac{-\ln x}{4}$$

The general solution

$$y = c_1 x^2 + c_2 x^{-2} - \frac{\ln x}{4}$$