March 8 Math 3260 sec. 55 Spring 2018

Section 4.2: Null & Column Spaces, Linear Transformations

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted[1] by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

$$\text{Nul}\, A = \{\mathbf{x} \overset{\text{in}}{\in} \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

We can say that Nul A is the subset of $\mathbb{R}^n$ that gets mapped to the zero vector under the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

[1] Some authors will write Null(A)—I tend to write two ells.

Example

Determine Nul A where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}.$$

The vectors $\vec{x}$ in Nul A satisfy $A\vec{x} = \vec{0}$.

$$A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \qquad \begin{aligned} x_1 &= -3x_3 \\ x_2 &= -2x_3 \\ x_3 &- \text{free} \end{aligned}$$

so

$$\vec{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix},$$

we can say $\text{Nul}\, A = \text{Span} \left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}$

Theorem

For $m \times n$ matrix A, Nul A is a subspace of $\mathbb{R}^n$.

We can show that Nul A contains the zero vector and is closed under the two operations.

Note, $A\vec{0} = \vec{0}$ hence $\vec{0}$ is in Nul A

in $\mathbb{R}^n$

If $\vec{u}$ and $\vec{v}$ are in Nul A, then $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$.

So

$$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$$

$\vec{u}+\vec{v}$ solves the homogeneous equation, so $\vec{u}+\vec{v}$ is in Nul A. Nul A is closed under vector addition.

Similarly, for scalar c

$$A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}$$

Hence $c\vec{u}$ is in Nul A making it closed under scalar multiplication.

Nul A is therefore a subspace of $\mathbb{R}^n$.

Example

For a given matrix, a spanning set for Nul$A$ gives an *explicit* description of this subspace. Find a spanning set for Nul A where

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 2 & 6 & -5 \end{bmatrix}.$$

Using the rref $\quad A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix}$

$x_1 = -2x_3 + x_4$

$x_2 = -2x_3 + 2x_4$

x_3, x_4 - free

so for $\vec{x}$ in Nul$A$

$$\vec{x} = \begin{bmatrix} -2x_3 + x_4 \\ -2x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_3 \\ -2x_3 \\ x_3 \\ 0 \end{bmatrix} + \begin{bmatrix} x_4 \\ 2x_4 \\ 0 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

A explicit description of Nul A is

$$\text{Nul}\,A = \text{Span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Column Space

Definition: The **column space** of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\mathrm{Col}A = \mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Note that this corresponds to the set of $\mathbf{b}$ for which linear equations of the form $A\mathbf{x} = \mathbf{b}$ are consistent! That is

$$\mathrm{Col}A = \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

m

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

This follow from the previous theorem that says a span is always a subspace.

Corollary: Col $A = \mathbb{R}^m$ if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Example

Find a matrix A such that $W = \text{Col}\, A$ where

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

We try to write elements of W as linear combinations.

$$\begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} = \begin{bmatrix} 6a \\ a \\ -7a \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} = a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}}_{A} \begin{bmatrix} a \\ b \end{bmatrix}$$

$$A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$$

Example

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

(a) If Col A is a subspace of $\mathbb{R}^k$, what is k?

Columns are in $\mathbb{R}^3$ so $k = 3$

(b) If Nul A is a subspace of $\mathbb{R}^k$, what is k?

$A\vec{x} = \vec{0}$

3×4 $k \times 1$

$A\vec{x}$ defined requires $\vec{x}$ in $\mathbb{R}^4$

so $k = 4$

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

(c) Is $\mathbf{u}$ in Nul A? Could $\mathbf{u}$ be in Col A?

Check: is $A\vec{u} = \vec{0}$? $\quad A\vec{u} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \vec{0}$

No, $\vec{u}$ is not in Nul A.

$\vec{u}$ is not in Col A. Col A is a subspace of $\mathbb{R}^3$ while $\vec{u}$ is in $\mathbb{R}^4$.

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(c) Is **v** in Col A? Could **v** be in Nul A?

Check : Is $A\vec{x} = \vec{v}$ consistent?

not a pivot column

$$[A \ \vec{v}] \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & 0 & -30/17 \\ 0 & 0 & 0 & 1 & 1/17 \end{bmatrix}$$

$A\vec{x} = \vec{v}$ is consistent

Yes, $\vec{v}$ is in Col$A$

$A\vec{v}$ isn't defined since $\vec{v}$ is in $\mathbb{R}^3$, so no, $\vec{v}$ could not be in Nul A.

Linear Transformation

Definition: Let V and W be vector spaces. A linear transformation $T : V \longrightarrow W$ is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every $\mathbf{u}$ in V and scalar c.

Example

Let $C^1(\mathbb{R})$ denote the set of all real valued functions that are differentiable and $C^0(\mathbb{R})$ the set of all continuous real valued functions. Note that differentiation is a linear transformation. That is

$$D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad D(f) = f'$$

satisfies the two conditions in the previous definition.

From calculus, we know $\frac{d}{dx}(f(x)+g(x)) = f'(x)+g'(x)$

i.e. $D(f+g) = D(f) + D(g)$

Also $\frac{d}{dx}(cf(x)) = cf'(x)$ i.e. $D(cf) = cD(f)$

Characterize the subset of $C^1(\mathbb{R})$ such that $D(f) = 0$.

$D(f)=0$ requires $f'(x)=0$ for all x. $\Rightarrow$ $f(x)=c$ for constant c.

Range and Kernel

Definition: The **range** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. (The set of all images of elements of V.)

Definition: The **kernel** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors $\mathbf{x}$ in V such that $T(\mathbf{x}) = \mathbf{0}$. (The analog of the null space of a matrix.)

Theorem: Given linear transformation $T : V \longrightarrow W$, the range of T is a subspace of W and the kernel of T is a subspace of V.

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(a) Express the equation that a function y must satisfy if y is in the kernel of T.

To be in the kernel $T(y) = 0$.

This requires

$$\frac{dy}{dx} + \alpha y = 0$$

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(b) Show that for any scalar c, $y = ce^{-\alpha x}$ is in the kernel of T.

y is the kernel if $\frac{dy}{dx} + \alpha y = 0$.

$y = ce^{-\alpha x}$, $\frac{dy}{dx} = -\alpha c e^{-\alpha x}$

so $\frac{dy}{dx} + \alpha y = -\alpha c e^{-\alpha x} + \alpha c e^{-\alpha x} = 0$