March 8 Math 3260 sec. 56 Spring 2018

Section 4.2: Null & Column Spaces, Linear Transformations

Definition: Let A be an $m \times n$ matrix. The **null space** of A, denoted[1] by Nul A, is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$. That is

in

$$\text{Nul}\, A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}.$$

We can say that Nul A is the subset of $\mathbb{R}^n$ that gets mapped to the zero vector under the linear transformation $\mathbf{x} \mapsto A\mathbf{x}$.

[1] Some authors will write Null(A)—I tend to write two ells.

Example

Determine Nul A where

$$A = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 2 & 7 \end{bmatrix}.$$

We need to characterize vectors $\vec{x}$ such that $A\vec{x} = \vec{0}$.

We can use the rref $\quad A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 2 \end{bmatrix} \quad$ $x_1 = -3x_3$

$x_2 = -2x_3$

x_3 - free

So for $\vec{x}$ in Nul A

$$\vec{x} = \begin{bmatrix} -3x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix}$$

we can write

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

Theorem

For $m \times n$ matrix A, Nul A is a subspace of $\mathbb{R}^n$.

We can show that Nul A satisfies the three subspace conditions: Contains $\vec{0}$, is closed under vector addition and closed under scalar multiplication.

① Note $A\vec{0} = \vec{0}$, so $\vec{0}$ is in Nul A.

② Suppose $\vec{u}$ and $\vec{v}$ are in Nul A. Then $A\vec{u} = \vec{0}$ and $A\vec{v} = \vec{0}$.

so
$$A(\vec{u}+\vec{v}) = A\vec{u} + A\vec{v} = \vec{0} + \vec{0} = \vec{0}$$

We see that $\vec{u}+\vec{v}$ is in Nul A making Nul A

Closed under vector addition.

(3) Consider $c\vec{u}$ for any scaler c.

$$A(c\vec{u}) = cA\vec{u} = c\vec{0} = \vec{0}.$$

Hence $c\vec{u}$ is in Nul A. Nul A is closed under scaler multiplication.

Nul A is a subspace of $\mathbb{R}^n$.

Example

For a given matrix, a spanning set for Nul$A$ gives an *explicit* description of this subspace. Find a spanning set for Nul A where

$$A = \begin{bmatrix} 1 & 0 & 2 & -1 \\ 1 & 2 & 6 & -5 \end{bmatrix}.$$

We can use the rref

$$A \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 2 & -1 \\ 0 & 1 & 2 & -2 \end{bmatrix}$$

$x_1 = -2x_3 + x_4$

$x_2 = -2x_3 + 2x_4$

x_3, x_4 - free

so $\vec{x}$ in Nul A

$$\vec{x} = \begin{bmatrix} -2x_3 + x_4 \\ -2x_3 + 2x_4 \\ x_3 \\ x_4 \end{bmatrix} = x_3 \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

So an explicit description of Nul A is

$$\text{Nul } A = \text{Span}\left\{ \begin{bmatrix} -2 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Column Space

Definition: The **column space** of an $m \times n$ matrix A, denoted Col A, is the set of all linear combinations of the columns of A. If $A = [\mathbf{a}_1 \quad \cdots \quad \mathbf{a}_n]$, then

$$\mathrm{Col} A = \mathrm{Span}\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}.$$

Note that this corresponds to the set of **b** for which linear equations of the form $A\mathbf{x} = \mathbf{b}$ are consistent! That is

$$\mathrm{Col} A = \{\mathbf{b} \in \mathbb{R}^m \mid \mathbf{b} = A\mathbf{x} \text{ for some } \mathbf{x} \in \mathbb{R}^n\}.$$

Theorem

The column space of an $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

Follows from our theorem on the

Span of a set of vectors.

Corollary: Col $A = \mathbb{R}^m$if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$ in $\mathbb{R}^m$.

Example

Find a matrix A such that $W = \text{Col } A$ where

$$W = \left\{ \begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} \mid a, b \in \mathbb{R} \right\}.$$

We want to write elements of W as a linear combination of vectors. Then we form a matrix with those vectors.

$$\begin{bmatrix} 6a - b \\ a + b \\ -7a \end{bmatrix} = \begin{bmatrix} 6a \\ a \\ -7a \end{bmatrix} + \begin{bmatrix} -b \\ b \\ 0 \end{bmatrix} = a \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}}_{\text{an } A} \begin{bmatrix} a \\ b \end{bmatrix}$$

So a matrix A is $\begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}$.

$$W = \text{Col}\, A$$

Example

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}$$

$A\vec{x} = \vec{b}$

(a) If Col A is a subspace of $\mathbb{R}^k$, what is k?

each column is in $\mathbb{R}^3$, so $k = 3$

$A\vec{x} = \vec{0}$

(b) If Nul A is a subspace of $\mathbb{R}^k$, what is k?

$A\vec{x}$ requires $\vec{x}$ in $\mathbb{R}^4$ so $k = 4$.

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{u} = \begin{bmatrix} 3 \\ -2 \\ -1 \\ 0 \end{bmatrix}$$

(c) Is $\mathbf{u}$ in Nul A? Could $\mathbf{u}$ be in Col A?

is $A\vec{u} = \vec{0}$?

$A\vec{u} = \begin{bmatrix} 0 \\ -3 \\ 3 \end{bmatrix} \neq \vec{0}$, so no, $\vec{u}$ is not in Nul A

$\vec{u}$ could not be in Col A. Col A is a subspace of $\mathbb{R}^3$ but $\vec{u}$ is in $\mathbb{R}^4$.

Example Continued...

$$A = \begin{bmatrix} 2 & 4 & -2 & 1 \\ -2 & -5 & 7 & 3 \\ 3 & 7 & -8 & 6 \end{bmatrix}, \quad \text{and} \quad \mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}$$

(c) Is **v** in Col A? Could **v** be in Nul A?

Is $A\vec{x} = \vec{v}$ consistent?

not a pivot column

$$\begin{bmatrix} A & \vec{v} \end{bmatrix} \xrightarrow{\text{rref}} \begin{bmatrix} 1 & 0 & 9 & 0 & 5 \\ 0 & 1 & -5 & 0 & -30/17 \\ 0 & 0 & 0 & 1 & 1/17 \end{bmatrix}$$

The system is consistent, $\vec{v}$ is in Col A.

$\vec{v}$ could not be in Nul A, $A\vec{v}$ is not defined since A has 4 columns and $\vec{v}$ is in $\mathbb{R}^3$.

Linear Transformation

Definition: Let V and W be vector spaces. A linear transformation $T : V \longrightarrow W$ is a rule that assigns to each vector $\mathbf{x}$ in V a unique vector $T(\mathbf{x})$ in W such that

(i) $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for every $\mathbf{u}, \mathbf{v}$ in V, and

(ii) $T(c\mathbf{u}) = cT(\mathbf{u})$ for every $\mathbf{u}$ in V and scalar c.

Example

Let $C^1(\mathbb{R})$ denote the set of all real valued functions that are differentiable and $C^0(\mathbb{R})$ the set of all continuous real valued functions. Note that differentiation is a linear transformation. That is

$$D : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R}), \quad D(f) = f'$$

satisfies the two conditions in the previous definition.

Recall $\frac{d}{dx}\left(f(x)+g(x)\right) = f'(x)+g'(x)$ i.e. $D(f+g) = D(f)+D(g)$

and $\frac{d}{dx}\left(cf(x)\right) = cf'(x)$ i.e. $D(cf) = cD(f)$

Characterize the subset of $C^1(\mathbb{R})$ such that $D(f) = 0$.

Such f satisfies $f'(x)=0$ for all x. These are constant functions.

Range and Kernel

Definition: The **range** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors in W of the form $T(\mathbf{x})$ for some $\mathbf{x}$ in V. (The set of all images of elements of V.)

Definition: The **kernel** of a linear transformation $T : V \longrightarrow W$ is the set of all vectors $\mathbf{x}$ in V such that $T(\mathbf{x}) = \mathbf{0}$. (The analog of the null space of a matrix.)

Theorem: Given linear transformation $T : V \longrightarrow W$, the range of T is a subspace of W and the kernel of T is a subspace of V.

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(a) Express the equation that a function y must satisfy if y is in the kernel of T.

y in kernel of T requires $T(y) = 0$.

This requires

$$\frac{dy}{dx} + \alpha y = 0$$

Example

Consider $T : C^1(\mathbb{R}) \longrightarrow C^0(\mathbb{R})$ defined by

$$T(f) = \frac{df}{dx} + \alpha f(x), \quad \alpha \text{ a fixed constant.}$$

(b) Show that for any scalar c, $y = ce^{-\alpha x}$ is in the kernel of T.

We need $\frac{dy}{dx} + \alpha y = 0$

If $y = ce^{-\alpha x}$, then $\frac{dy}{dx} = -\alpha c e^{-\alpha x}$

$$\frac{dy}{dx} + \alpha y = -\alpha c e^{-\alpha x} + \alpha c e^{-\alpha x} = 0$$

So y is in the kernel of T.