March 9 Math 1190 sec. 62 Spring 2017

Section 3.3: Derivatives of Logarithmic Functions

We have the new rules:

$$\frac{d}{dx}\log_a(x) = \frac{1}{x\ln a}, \qquad \frac{d}{dx}\ln(x) = \frac{1}{x}$$

with the chain rule

$$\frac{d}{dx}\log_a(f(x)) = \frac{f'(x)}{f(x)\ln a}, \qquad \frac{d}{dx}\ln(f(x)) = \frac{f'(x)}{f(x)}$$

Properties of Logs

We'll make use of the properties

$$\log_a(xy) = \log_a x + \log_a y$$

$$\log_a\left(\frac{x}{y}\right) = \log_a x - \log_a y$$

$$\log_a(x^r) = r\log_a x$$

for $x, y > 0$, and any base $a > 0$, $a \neq 1$.

Using Properties of Logs

Properties of logarithms can be used to simplify expressions characterized by products, quotients and powers.

Illustrative Example: Evaluate $\displaystyle \frac{d}{dx}\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right)$

First we rewrote the function using properties of logs

$$\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = 2\ln x + \ln\cos(2x) - \frac{1}{3}\ln(x^2+x).$$

Then we took the derivative of the sum

$$\frac{d}{dx}\ln\left(\frac{x^2\cos(2x)}{\sqrt[3]{x^2+x}}\right) = \frac{2}{x} - \frac{2\sin(2x)}{\cos(2x)} - \frac{1}{3}\frac{2x+1}{x^2+x}.$$

Question

Expand the following **completely** as a sum/difference/and multiple of logs

$$\ln\left(\frac{x^3+4}{\sqrt{x}\tan x}\right)$$

$= \ln(x^3+4) - \ln(\sqrt{x}\tan x)$

(a) $3\ln x + \ln 4 - \ln\sqrt{x} - \ln\tan x$

$= \ln(x^3+4) - \left(\ln x^{1/2} + \ln\tan x\right)$

(b) $\ln(x^3+4) - \ln\sqrt{x} - \ln\tan x$

$= \ln(x^3+4) - \ln x^{1/2} - \ln\tan x$

(c) $\ln(x^3+4) - \frac{1}{2}\ln x + \ln\tan x$

$= \ln(x^3+4) - \frac{1}{2}\ln x - \ln\tan x$

(d) $\ln(x^3+4) - \frac{1}{2}\ln x - \ln\tan x$

Question

Evaluate the derivative.

$$\frac{d}{dx}\ln\left(\frac{x^3+4}{\sqrt{x}\tan x}\right)$$

$$= \frac{d}{dx}\ln(x^3+4) - \frac{d}{dx}\frac{1}{2}\ln x - \frac{d}{dx}\ln\tan x$$

(a) $\dfrac{3x^2}{x^2}+\dfrac{1}{4}+\dfrac{1}{2x}+\dfrac{\sec^2 x}{\tan x}$

(b) $\dfrac{3x^2}{x^3+4}-\dfrac{1}{2x}+\dfrac{\sec^2 x}{\tan x}$

(c) $\dfrac{3x^2}{x^3+4}-\dfrac{1}{2x}-\dfrac{\sec^2 x}{\tan x}$

Logarithmic Differentiation

We can use properties of logarithms to simplify the process of taking derivatives of expressions that are complicated by

products quotients and powers.

Illustrative Example: Evaluate $\displaystyle \frac{d}{dx}\left(\frac{x^2\sqrt{x+1}}{\cos^4(3x)}\right)$

Let $y = \dfrac{x^2\sqrt{x+1}}{\cos^4(3x)}$. Instead of finding $\frac{dy}{dx}$ directly, we'll take the natural log of y and use implicit diff.

$$\ln y = \ln\left(\frac{x^2\sqrt{x+1}}{\cos^4(3x)}\right)$$

Expand using log properties

$$\ln y = \ln\left(x^2\sqrt{x+1}\right) - \ln\left(\cos^4(3x)\right)$$

$$= \ln x^2 + \ln(x+1)^{1/2} - \ln\left(\left(\cos(3x)\right)^4\right)$$

$$\ln y = 2\ln x + \frac{1}{2}\ln(x+1) - 4\ln\cos(3x)$$

Now take $\frac{d}{dx}$ of both sides

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(2\ln x + \frac{1}{2}\ln(x+1) - 4\ln\cos(3x)\right)$$

$$\frac{\frac{dy}{dx}}{y} = 2\,\frac{1}{x} + \frac{1}{2}\,\frac{1}{x+1} - 4\,\frac{(-\sin(3x)\cdot 3)}{\cos(3x)}$$

$$\frac{\frac{dy}{dx}}{y} = \frac{2}{x} + \frac{1}{2}\frac{1}{x+1} + 12\frac{\sin(3x)}{\cos(3x)}$$

Since $y = \frac{x^2\sqrt{x+1}}{\cos^4(3x)}$, $\frac{dy}{dx}$ is the derivative we're looking for.

Isolate $\frac{dy}{dx}$

$$\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{1}{2}\frac{1}{x+1} + 12\tan(3x)\right)$$

Substitute $y = \frac{x^2\sqrt{x+1}}{\cos^4(3x)}$

$$\boxed{\frac{dy}{dx} = \frac{x^2\sqrt{x+1}}{\cos^4(3x)}\left(\frac{2}{x} + \frac{1}{2}\frac{1}{x+1} + 12\tan(3x)\right)}$$

Logarithmic Differentiation

If the differentiable function $y = f(x)$ consists of complicated products, quotients, and powers:

(i) Take the logarithm of both sides, i.e. $\ln(y) = \ln(f(x))$. Then use properties of logs to express $\ln(f(x))$ as a sum/difference of simpler terms.

(ii) Take the derivative of each side, and use the fact that $\frac{d}{dx}\ln(y) = \frac{\frac{dy}{dx}}{y}$.

(iii) Solve for $\frac{dy}{dx}$ (i.e. multiply through by y), and replace y with $f(x)$ to express the derivative explicitly as a function of x.

Example

Find $\frac{dy}{dx}$.

$$y = x^{\tan x}$$

This is neither exponential nor power. It's complicated by a variable base and variable power.

Using log diff.

$$\ln y = \ln x^{\tan x}$$ ← take the log

$$\ln y = \tan x \ln x$$ ← use log properties

tan x times ln x is a product.

$$\frac{d}{dx} \ln y = \frac{d}{dx}\left(\tan x \ln x\right)$$ ← take the derivative

product rule

$$\frac{\frac{dy}{dx}}{y} = \sec^2 x \ln x + \tan x \left(\frac{1}{x}\right)$$

$$\frac{dy}{dx} = y\left(\sec^2 x \ln x + \frac{\tan x}{x}\right)$$ ← mult. by y

$$\frac{dy}{dx} = x^{\tan x}\left(\sec^2 x \ln x + \frac{\tan x}{x}\right)$$ ← sub in $y = x^{\tan x}$

Example

Find $\frac{dy}{dx}$.

$$y = \frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}$$

We'll use log. diff.

$$\ln y = \ln\left(\frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}\right)$$

$$= \ln\left(x^3(4x-1)^5\right) - \ln(x+5)^{1/4}$$

$$= \ln x^3 + \ln(4x-1)^5 - \ln(x+5)^{1/4}$$

$$= 3\ln x + 5\ln(4x-1) - \frac{1}{4}\ln(x+5)$$

$$\frac{d}{dx}\ln y = \frac{d}{dx}\left(3\ln x + 5\ln(4x-1) - \frac{1}{4}\ln(x+5)\right)$$

$$\frac{y'}{y} = 3\,\frac{1}{x} + 5\,\frac{4}{4x-1} - \frac{1}{4}\,\frac{1}{x+5}$$

$$\frac{y'}{y} = \frac{3}{x} + \frac{20}{4x-1} - \frac{1}{4}\,\frac{1}{x+5}$$

$$y' = y\left(\frac{3}{x} + \frac{20}{4x-1} - \frac{1}{4}\,\frac{1}{x+5}\right)$$

$$\boxed{y' = \frac{x^3(4x-1)^5}{\sqrt[4]{x+5}}\left(\frac{3}{x} + \frac{20}{4x-1} - \frac{1}{4}\,\frac{1}{x+5}\right)}$$

Question

Find $\frac{dy}{dx}$.

$$y = \sqrt[3]{x^2 \sin(x)}$$

$$\ln y = \ln \sqrt[3]{x^2 \sin(x)}$$

$$= \ln \left(x^2 \sin(x)\right)^{1/3}$$

$$= \frac{1}{3} \ln \left(x^2 \sin(x)\right)$$

$$= \frac{1}{3} \left(\ln x^2 + \ln \sin(x)\right)$$

(a) $\frac{dy}{dx} = \left[\sqrt[3]{x^2 \sin(x)}\right] \left(\frac{2}{x} + \cot x\right)$

$$\frac{d}{dx} \ln x^2 = \frac{2x}{x^2} = \frac{2}{x}$$

(b) $\frac{dy}{dx} = \left[\sqrt[3]{x^2 \sin(x)}\right] \left(\frac{2}{3x} + \frac{1}{3} \cot x\right)$

$$\frac{d}{dx} \ln \sin x = \frac{\cos x}{\sin x}$$

$$= \cot x$$

(c) $\frac{dy}{dx} = \left[\sqrt[3]{x^2 \sin(x)}\right] \left(\frac{1}{x^2} + \frac{1}{\sin x}\right)$

Section 4.5: Indeterminate Forms & L'Hôpital's Rule

Consider the following three limit statements (all of which are true):

(a) $$\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$$

(b) $$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

(c) $$\lim_{x\to 3} \frac{x^2-9}{(x-3)^2} \quad \text{doesn't exist}$$

Note: Each of these three limits involve both numerator and denominator going to zero—giving the form $\frac{0}{0}$. In the top two, the limit exists, but the limits are different. In the third, the limit doesn't exist.

Indeterminate Forms

$0/0$ is called an **Indeterminate form**.

Other indeterminate forms we'll encounter include

$$\frac{\pm\infty}{\pm\infty}, \quad \infty-\infty, \quad 0\cdot\infty, \quad 1^{\infty}, \quad 0^0, \quad \text{and} \quad \infty^0.$$

Indeterminate forms are not defined (as numbers)

Question

(1) **True or False:** $\infty - \infty = 0$.

$\infty - \infty$ is not defined

(2) **True or False:** The form $\frac{1}{0}$ is indeterminate.

This is either an ∞ or not existing

(3) **True or False:** $\frac{0}{1} = 0$.

Theorem: l'Hospital's Rule (part 1)

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x\to c} f(x) = 0 \quad \text{and} \quad \lim_{x\to c} g(x) = 0$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

This is useful (perhaps) when the limit

$\lim_{x\to c} \frac{f(x)}{g(x)}$ gives the form "$\frac{0}{0}$"

Evaluate each limit if possible

(a) $\displaystyle\lim_{x\to1}\frac{\ln x}{x-1}$ $=$ " $\frac{0}{0}$ "

Note $\displaystyle\lim_{x\to1}\ln x = \ln 1 = 0$

and $\displaystyle\lim_{x\to1} x-1 = 1-1 = 0$

apply l'H rule

$$\lim_{x\to1}\frac{\ln x}{x-1} = \lim_{x\to1}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}(x-1)}$$

$$= \lim_{x\to1}\frac{\frac{1}{x}}{1} = \frac{\frac{1}{1}}{1} = 1$$

Theorem: l'Hospital's Rule (part 2)

Suppose f and g are differentiable on an open interval I containing c (except possibly at c), and suppose $g'(x) \neq 0$ on I. If

$$\lim_{x\to c} f(x) = \pm\infty \quad \text{and} \quad \lim_{x\to c} g(x) = \pm\infty$$

then

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \lim_{x\to c} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists (or is ∞ or $-\infty$).

We can use this if

$$\lim_{x\to c} \frac{f(x)}{g(x)} = \text{"}\frac{\pm\infty}{\pm\infty}\text{"}$$

Recall $\lim_{x\to\infty} e^{-x} = 0$

(b) $\lim_{x\to\infty} xe^{-x}$ = "$\infty \cdot 0$"

$\infty \cdot 0$ is an indeterminate form, but it's not $\frac{0}{0}$ or $\frac{\infty}{\infty}$

We need to write xe^{-x} as a quotient.

we can write $xe^{-x} = \dfrac{e^{-x}}{\frac{1}{x}}$ or $xe^{-x} = \dfrac{x}{\frac{1}{e^{-x}}}$

We'll use the second one since $\dfrac{1}{e^{-x}} = e^{x}$

So $xe^{-x} = \dfrac{x}{e^{x}}$

$$\lim_{x\to\infty} x e^{-x} = \lim_{x\to\infty} \frac{x}{e^x} = "\frac{\infty}{\infty}"$$

$$\lim_{x\to\infty} e^x = \infty$$

Now l'H rule applies

Using l'H rule

$$\lim_{x\to\infty} \frac{x}{e^x} = \lim_{x\to\infty} \frac{\frac{d}{dx}x}{\frac{d}{dx}e^x} = \lim_{x\to\infty} \frac{1}{e^x} = 0$$

Remember $"\frac{k}{\infty}"$ for constant k is zero.

(c) $\lim_{x \to 0} \frac{\cos x - 1}{x^2}$ $=$ "$\frac{0}{0}$"

$\lim_{x \to 0} \cos x - 1 = \cos 0 - 1 = 1 - 1 = 0$

Use l'H rule

$= \lim_{x \to 0} \frac{\frac{d}{dx}(\cos x - 1)}{\frac{d}{dx} x^2}$

$= \lim_{x \to 0} \frac{-\sin x}{2x} =$ "$\frac{0}{0}$"

we can use the rule again since we have a form it applies to!

apply l'H again

$= \lim_{x \to 0} \frac{\frac{d}{dx}(-\sin x)}{\frac{d}{dx}(2x)}$

$$= \lim_{x \to 0} \frac{-\cos x}{2} = \frac{-\cos 0}{2} = \frac{-1}{2}$$