Mar. 9 Math 2254H sec 015H Spring 2015

Section 10.3: Polar Coordinates

Converting between Coordinate Systems

The coordinates (x, y) are called **rectangular** or **Cartesian** coordinates.

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$x^2 + y^2 = r^2, \quad \tan\theta = \frac{y}{x} \quad \text{for } x \neq 0$$

If $x = 0$, then $\theta = \frac{\pi}{2}$ or $\theta = -\frac{\pi}{2}$—of course any co-terminal θ may be used.

Polar Graphs

An analog to $y = f(x)$ is $r = f(\theta)$. Converting to Cartesian may or may not be useful. For example

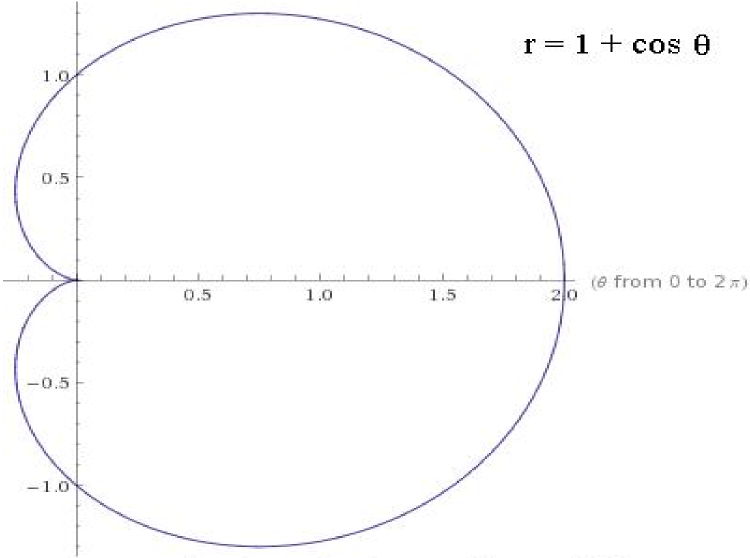
$$r = 4\sin\theta \quad \Longleftrightarrow \quad x^2 + (y-2)^2 = 2^2$$

This obviously gives a circle with radius 2 centered at $(0,2)$.

$$r = 1 + \cos\theta \quad \Longleftrightarrow \quad (x^2+y^2)^2 - 2x^3 - 2xy^2 - y^2 = 0$$

It's not particularly obvious what kind of graph this is.

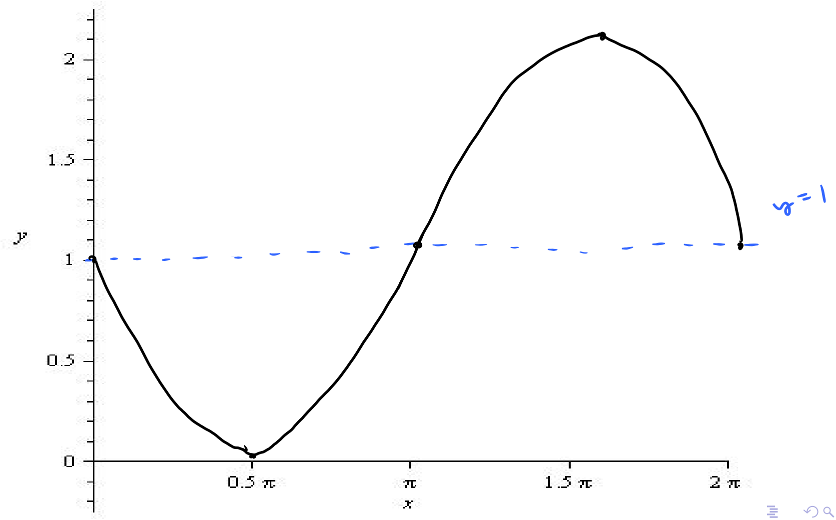
A plot of the *cardioid* $r = 1 + \cos\theta$



(Produced using Wolfram Alpha)

Producing a Polar Graph

Plot the graph of $y = 1 - \sin x$ for $0 \leq x \leq 2\pi$ in **Cartesian** coordinates.



Producing a Polar Graph

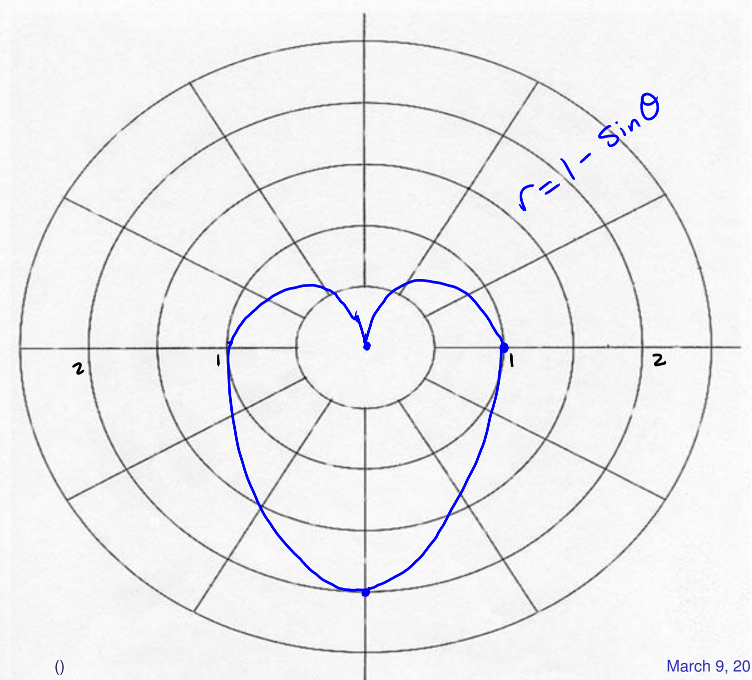
Analyze the graph and consider a few points for the polar equation $r = 1 - \sin\theta$, and produce a polar plot on the following side.

As $0 \le \theta \le \frac{\pi}{2}$ r goes from 1 to the origin

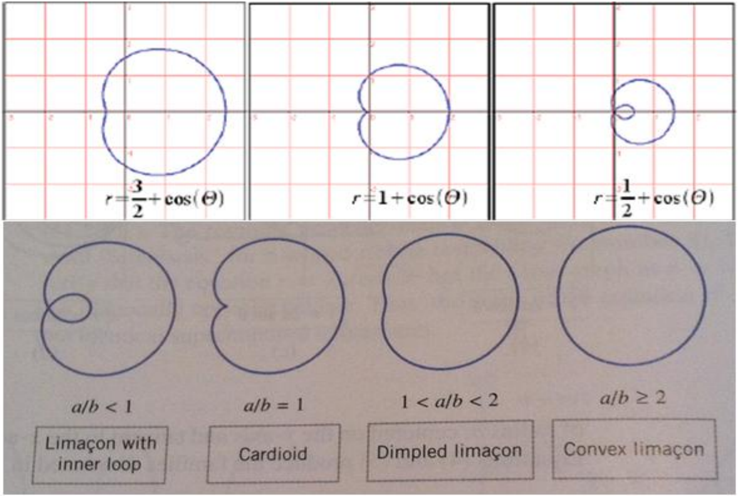
$\frac{\pi}{2} \le \theta \le \pi$ r goes from 0 back to 1

$\pi \le \theta \le \frac{3\pi}{2}$ r goes from 1 to 2

$\frac{3\pi}{2} \le \theta \le 2\pi$ r decreases from 2 to 1.

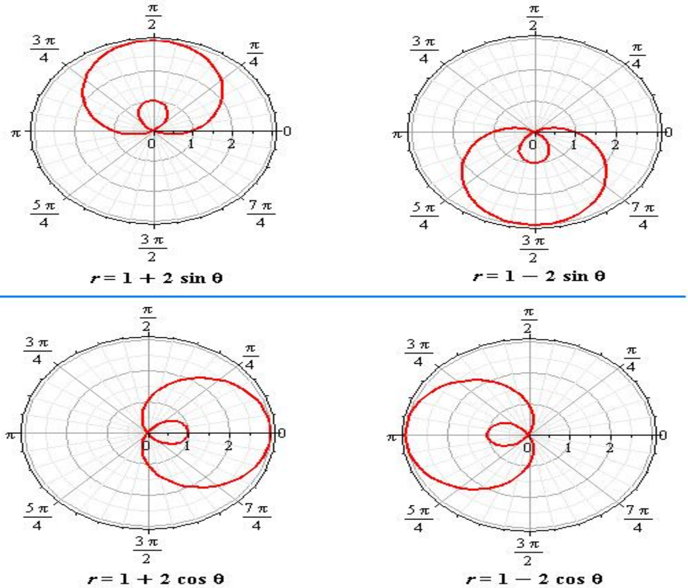

$r = 1 - \sin\theta$
2
1
1
2

Limaçons: $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$



$$r = a + b\cos\theta$$

Limaçons: $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$



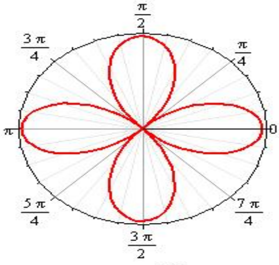
Roses $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$, $n = 2, 3, 4...$

The polar graph of the curve

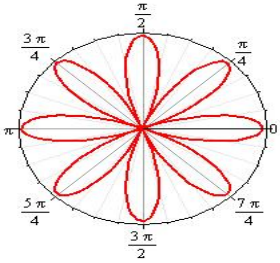
$$r = a\cos(n\theta) \quad \text{or} \quad r = a\sin(n\theta) \quad \text{for positive integer } n.$$

is a/an $\begin{cases} n \text{ petal rose,} & \text{if } n \text{ is odd} \\ 2n \text{ petal rose,} & \text{if } n \text{ is even} \end{cases}$

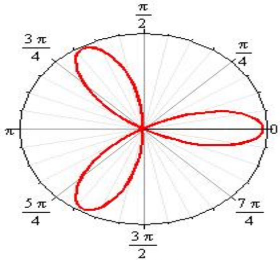
Roses $r = a\cos(n\theta)$ or $r = a\sin(n\theta)$, $n = 2, 3, 4...$



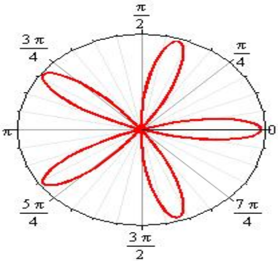
$r = \cos 2\theta$



$r = \cos 4\theta$



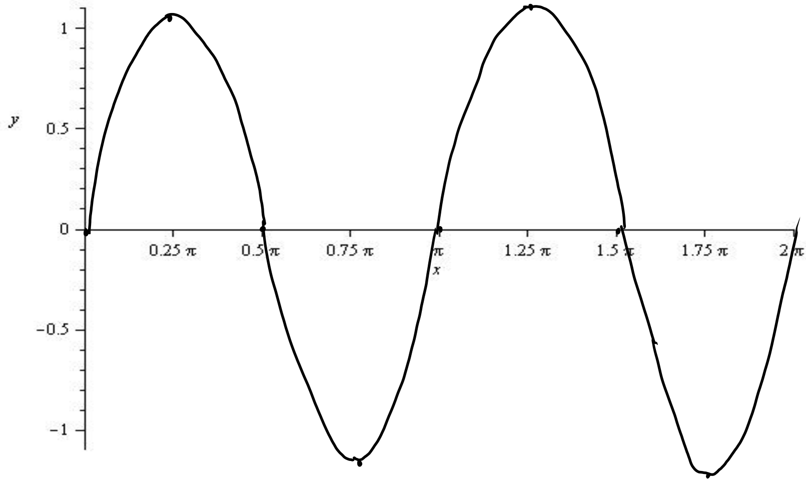
$r = \cos 3\theta$



$r = \cos 5\theta$

Example: Producing a 4 Petal Rose

Plot the graph of $y = \sin(2x)$ for $0 \leq x \leq 2\pi$ in **Cartesian coordinates.**



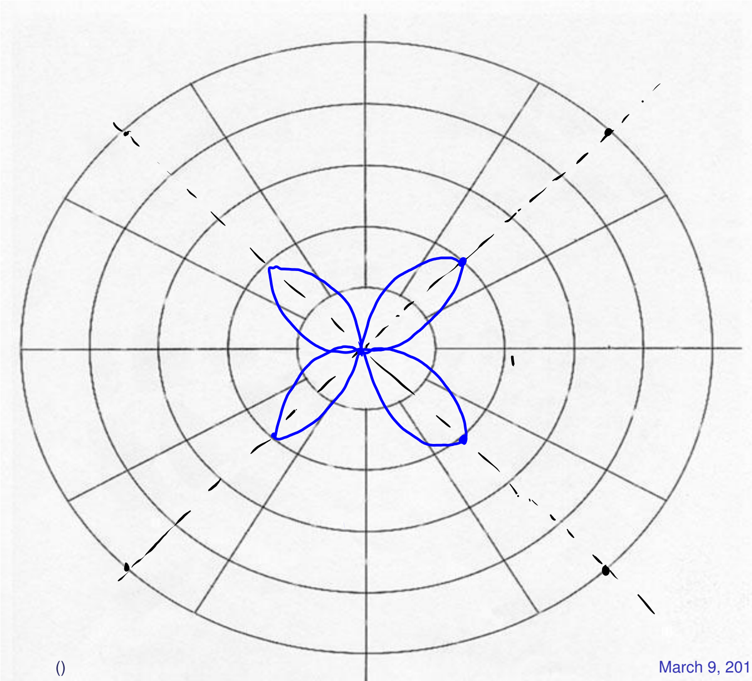
Example: Producing a 4 Petal Rose

Analyze the Caresian graph and find a few points on the curve. Use this information to produce a polar plot of $r = \sin(2\theta)$ on the next slide.

Major features occur in intervals of length $\frac{\pi}{4}$.

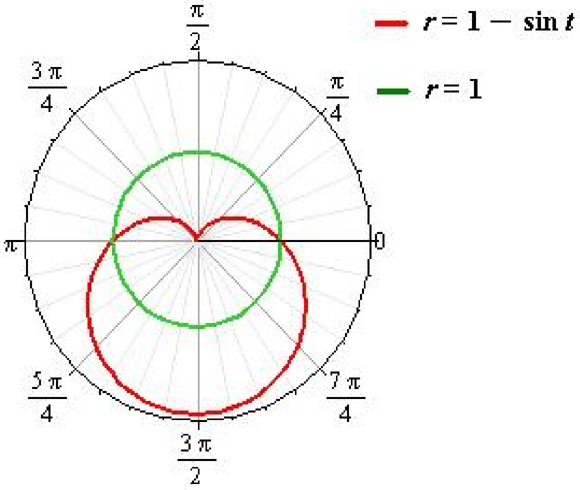
when $r < 0$, recall that points are reflected through the origin.

i.e. for $\frac{\pi}{2} < \theta < \frac{3\pi}{4}$ (quad II) the graph is in quad IV.

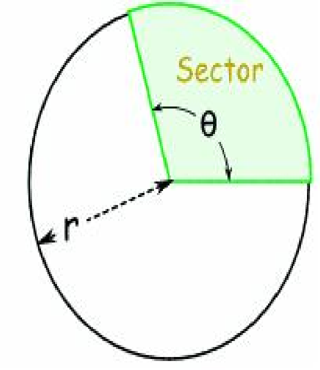


Section 5.4: Areas in Polar Coordinates

Motivating Example: Suppose we wish to find the area inside the circle $r = 1$ and outside of the cardioid $r = 1 - \sin\theta$.



Area of a Sector of a Circle



Area Sector

$$A = \pi r^2 \cdot \left(\frac{\theta}{2\pi}\right)$$

$$= \frac{1}{2} r^2 \theta$$

Figure: Recall the formula for the area of a sector of a circle of radius r and central angle θ.

Area **Bounded by** a Polar Graph $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$

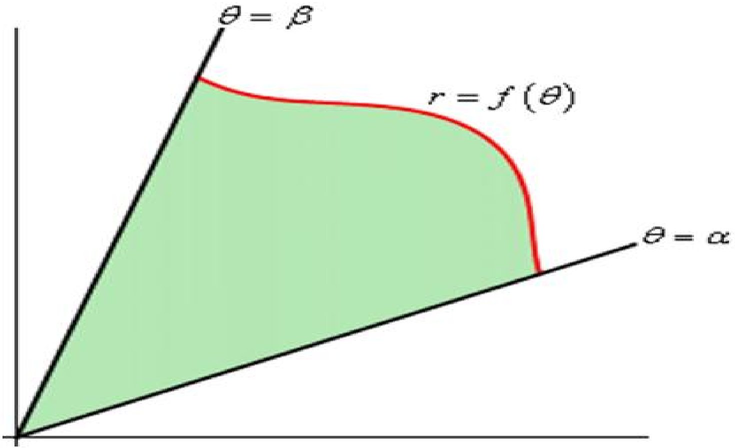


Figure: We *slice* the region into small wedges (a.k.a. form a **partition** of $[\alpha, \beta]$).

Area **Bounded by** a Polar Graph $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$

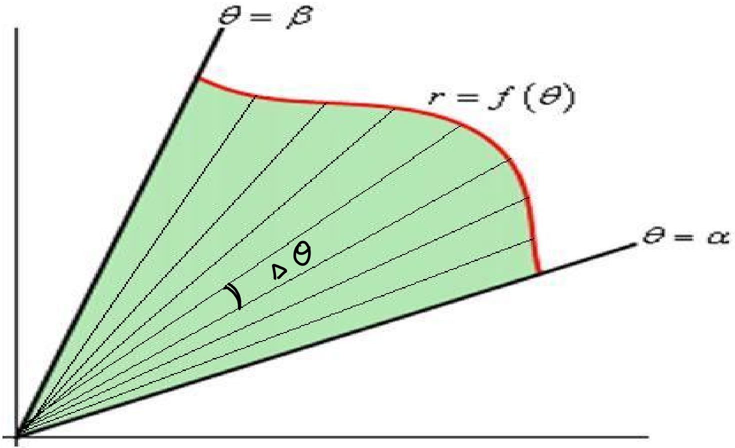
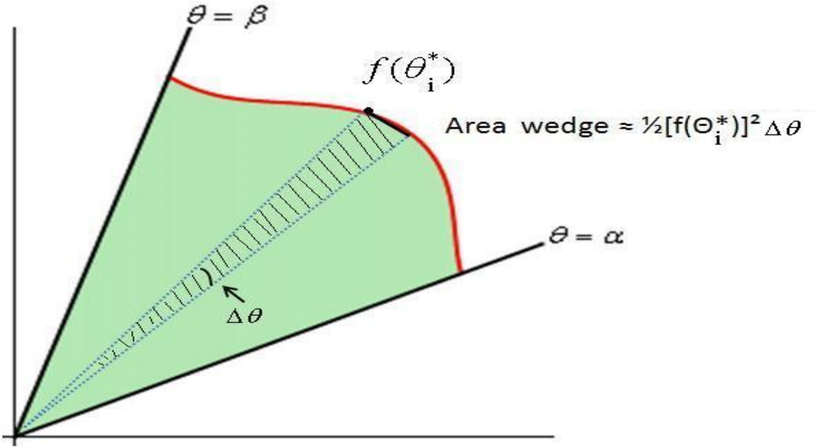


Figure: The region with a **partition**.

Area **Bounded by** a Polar Graph $r = f(\theta)$ for $\alpha \leq \theta \leq \beta$



Area **Bounded by** a Polar Graph $r = f(\theta)$ for $\alpha \le \theta \le \beta$

We add the areas of the individual wedges. The total area

$$A \approx \frac{1}{2}[f(\theta_1^*)]^2\Delta\theta + \cdots + \frac{1}{2}[f(\theta_n^*)]^2\Delta\theta = \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2\Delta\theta.$$

Take the limit as the number of wedges goes to ∞—that is, as $\Delta\theta$ becomes infinitesimal—to get

$$A = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{2}[f(\theta_i^*)]^2\Delta\theta = \frac{1}{2}\int_\alpha^\beta [f(\theta)]^2\, d\theta$$

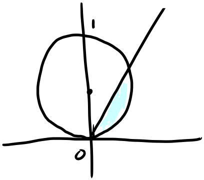
We can restate the formula as

$$A = \frac{1}{2}\int_\alpha^\beta r^2\, d\theta$$

Examples

Find the area of the region bounded by the curve $r = \sin\theta$ for $0 \le \theta \le \frac{\pi}{4}$.

$r = \sin\theta$



$$A = \int_0^{\pi/4} \frac{1}{2} r^2 \, d\theta$$

$$= \int_0^{\pi/4} \frac{1}{2} \sin^2\theta \, d\theta$$

$$= \frac{1}{2} \int_0^{\pi/4} \left(\frac{1}{2} - \frac{1}{2}\cos 2\theta\right) d\theta$$

$$= \frac{1}{2}\left(\frac{1}{2}\theta - \frac{1}{2}\cdot\frac{1}{2}\sin 2\theta\right)\Bigg|_0^{\pi/4}$$

$$= \frac{1}{2}\left(\frac{1}{2}\cdot\frac{\pi}{4} - \frac{1}{4}\sin\frac{\pi}{2} - 0\right)$$

$$= \frac{\pi}{16} - \frac{1}{8}$$