March 9 Math 3260 sec. 51 Spring 2020

Section 4.3: Linearly Independent Sets and Bases

Definition: Let H be a subspace of a vector space V. An indexed set of vectors $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_p\}$ in V is a **basis** of H provided

(i) $\mathcal{B}$ is linearly independent, and

(ii) $H =$Span$(\mathcal{B})$.

We can think of a basis as a *minimal spanning set*. All of the *information* needed to construct vectors in H is contained in the basis, and none of this information is repeated.

Example

If A is an invertible $n \times n$ matrix, then we know that (1) the columns are linearly independent, and (2) the columns span $\mathbb{R}^n$. Use this to determine if $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a basis for $\mathbb{R}^3$ where

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}.$$

We can create a matrix $A = [\vec{v}_1 \ \vec{v}_2 \ \vec{v}_3]$.

$$A = \begin{bmatrix} 3 & -4 & -2 \\ 0 & 1 & 1 \\ -6 & 7 & 5 \end{bmatrix}$$

We can take $\det(A)$.

$$\det(A) = \overset{=0}{a_{21}} C_{21} + a_{22} C_{22} + a_{23} C_{23}$$

$$= 1 \, (-1)^{2+2} \begin{vmatrix} 3 & -2 \\ -6 & 5 \end{vmatrix} + 1 \, (-1)^{2+3} \begin{vmatrix} 3 & -4 \\ -6 & 7 \end{vmatrix}$$

$$= (15 - 12) - 1 \, (21 - 24)$$

$$= 3 - (-3) = 6 \neq 0$$

A^{-1} exists, so the columns are linearly independent and span $\mathbb{R}^3$.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$.

Standard Basis in $\mathbb{R}^n$

The columns of the $n \times n$ identity matrix provide an obvious basis for $\mathbb{R}^n$. This is called the **standard basis** for $\mathbb{R}^n$. For example, the standard bases in $\mathbb{R}^2$ and $\mathbb{R}^3$ are

$$\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}, \quad \text{and} \quad \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \quad \text{respectively.}$$

Some Other Vector Spaces

- $\{1, t, t^2, t^3\}$ is the *standard basis* for $\mathbb{P}_3$
- The set $\{1, t, \ldots, t^n\}$ is called the *standard basis* for $\mathbb{P}_n$.
- The set $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is a basis for $M^{2\times 2}$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} + d\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Example

Define a *standard basis* for $M^{2\times 3}$.

$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix}$ a typical vector in $M^{2\times 3}$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Prelude to a Spanning Set Theorem

Example: Let $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ be vectors in a vector space V, and suppose that

(1) $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and
(2) $\mathbf{v}_3 = \mathbf{v}_1 - 2\mathbf{v}_2$.

Show that $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$.

Let $\vec{u}$ be in H, i.e. $\vec{u}$ is in Span$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$.

$$\vec{u} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3.$$

$$= c_1\vec{v}_1 + c_2\vec{v}_2 + c_3(\vec{v}_1 - 2\vec{v}_2)$$

$$= (c_1 + c_3)\vec{v}_1 + (c_2 - 2c_3)\vec{v}_2$$

So $\vec{u}$ is in $\text{Span}\{\vec{v}_1, \vec{v}_2\}$.

This can be done with every vector in H, so $H = \text{Span}\{\vec{v}_1, \vec{v}_2\}$.

Theorem:

Let $S = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_p\}$ be a set in a vector space V and $H =$Span(S).

(a.) If one of the vectors in S, say $\mathbf{v}_k$ is a linear combination of the other vectors in S, then the subset of S obtained by eliminating $\mathbf{v}_k$ still spans H.

(b) If $H \neq \{\mathbf{0}\}$, then some subset of S is a basis for H.

If we start with a spanning set, we can eliminate *duplication* and arrive at a basis.

Column Space

Find a basis for the column space matrix B that is in reduced row echelon form

$$B = \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3 \ \vec{b}_4 \ \vec{b}_5]$$

Columns 1, 3, and 5 are pivot columns.

Note $\vec{b}_2 = 4\vec{b}_1$ and $\vec{b}_4 = 2\vec{b}_1 - \vec{b}_3$

$$\text{Col } B = \text{Span}\{\vec{b}_1, \vec{b}_2, \vec{b}_3, \vec{b}_4, \vec{b}_5\}.$$

(By definition)

By our theorem, we can remove b_2 and b_4

and still have a spanning set.

The pivot columns are linearly independent. A basis for Col B is $\{\vec{b}_1, \vec{b}_3, \vec{b}_5\}$

the set of pivot columns.

Using the rref

Theorem: If $A = [\mathbf{a}_1 \cdots \mathbf{a}_n]$ and $B = [\mathbf{b}_1 \cdots \mathbf{b}_n]$ are row equivalent matrices, then Nul $A =$ Nul B. That is, the equations

$$A\mathbf{x} = \mathbf{0} \quad \text{and} \quad B\mathbf{x} = \mathbf{0}$$

have the same solution set.

Note what this means! It means that $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ and $\{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ have **exactly the same linear dependence relationships**!

Theorem:

The pivot columns of a matrix A form a basis of Col A.

Caveat: This means we can use row reduction to identify a basis, but the vectors we obtain will be from the original matrix A. (As illustrated in the following example.)

Find a basis for Col A[1]

$$A = \begin{bmatrix} 1 & 4 & 0 & 2 & -1 \\ 3 & 12 & 1 & 5 & 5 \\ 2 & 8 & 1 & 3 & 2 \\ 5 & 20 & 2 & 8 & 8 \end{bmatrix}.$$

We just need to know which are the pivot columns.

$$\xrightarrow{\text{rref}} \begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The pivot columns are 1, 3, and 5.

A basis for Col A is

$$\{\vec{a}_1, \vec{a}_3, \vec{a}_5\} = \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 5 \\ 2 \\ 8 \end{bmatrix} \right\}$$

[1] Use a calculator to do the row reduction.

Find bases for Nul A and Col A

$$A = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 2 & 1 & 5 & 1 \end{bmatrix}$$

we can use the rref for Both. For the null space we considering $A\vec{x} = \vec{0}$.

$$\text{rref}(A) = \begin{bmatrix} 1 & 0 & 3 & -2 \\ 0 & 1 & -1 & 5 \end{bmatrix}$$

If $A\vec{x} = \vec{0}$ then

$$x_1 = -3x_3 + 2x_4$$
$$x_2 = x_3 - 5x_4$$
$$x_3, x_4 - \text{free}$$

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -3x_3 + 2x_4 \\ x_3 - 5x_4 \\ x_3 \\ x_4 \end{bmatrix}$$

$$= x_3 \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

A basis for Nul A is $\left\{ \begin{bmatrix} -3 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -5 \\ 0 \\ 1 \end{bmatrix} \right\}$.

A basis for Col A is: $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

Example

Let $H = \{\mathbf{p} \text{ in } \mathbb{P}_2 \mid \mathbf{p}(-1) = 0\}$. Find a basis for H.

An element of $\mathbb{P}_2$ looks like

$\vec{p}(t) = p_0 + p_1 t + p_2 t^2$. For such $\vec{p}$ in H,

$$\vec{p}(-1) = p_0 + p_1(-1) + p_2(-1)^2 = 0$$

$$p_0 - p_1 + p_2 = 0$$

This looks like a homogeneous system

$p_0 = p_1 - p_2$, p_1, p_2 are free

If we set $p_1 = 1$ and $p_2 = 0$, we get

$\vec{p}_1(t) = 1 + 1t = 1 + t$

If we set $p_1 = 0$ and $p_2 = 1$, we get

$\vec{p}_2(t) = -1 + 1t^2 = -1 + t^2$

A basis for H is

$$\{1+t, -1+t^2\}$$